TECHNICAL REPORT 1969
September 2008

The Modal Decomposition of the Quality Factor of an Antenna in Prolate Spheroidal Coordinates

R. C. Adams
P. M. Hansen

SSC San Diego

SSC SAN DIEGO
San Diego, California 92152-5001

M. T. Kohlheim, CAPT, USN	**C. A. Keeney**
Commanding Officer	**Technical Director**

ADMINISTRATIVE INFORMATION

The work described in this report was performed for Space and Naval Warfare Systems Command Program Executive Office C4I (PMW 770-2) by the Electro Technology Branch (Code 55420), Space and Naval Warfare Systems Center San Diego.

Released under authority of
C. Milligan, Head
Electro Technology Branch

Released under authority of
S. D. Russell, Head
Electromagnetics &
Advanced Technology
Division

TECHNICAL REPORT 1969
September 2008

The Modal Decomposition of the Quality Factor of an Antenna in Prolate Spheroidal Coordinates

R. C. Adams
P. M. Hansen

SSC San Diego
San Diego, CA 92152-5001

EXECUTIVE SUMMARY

In 2004, SPAWAR Systems Center San Diego (SSC San Diego) published a technical document, TD 3188, "Evaluation of Q in an Electrically Small Antenna in Prolate Spheroidal Coordinates," which was written by the authors of this report. This document presented results for the radiation quality factor (Q_r) of an antenna surrounded by a surface in the shape of a prolate spheroid. It was assumed that there was no energy within the surface and that the lowest mode dominated.

At the Institute of Electrical and Electronics Engineers (IEEE) Antennas and Propagation Symposium (APS) conference held in Washington, DC in 2005, the authors extended the results to oblate spheroids. For the lowest mode and extremely long wavelengths, they presented analytic formulas that supported numerical results that compared to Chu's result for a sphere to those of spheroids with the same height or same volume.

A separate paper at the same symposium compared the results for both a prolate and oblate spheroid to calculated Q_r for practical antennas. At both high and low aspect ratios, some practical antennas have Q_r less than that calculated for the lowest order spheroidal mode, an indication that the minimum Q_r for a spheroid must include higher order modes.

This report discusses extending the results for prolate spheroids by including higher order modes and finding the sum of modes giving the minimum radiation Q. There is no coupling between odd and even modes, but there is coupling between the lowest order even modes. The report presents a simple result for the values of the relative weights for mixing modes 2, 4, and 6 that minimizes Q_r. One finding was for long wavelength the relative excitations should be proportional to the square of the frequency for mode 2 and to the fourth power of the frequency for mode 4. The coefficients for modes 4 and 6 that minimize Q_r are numerically zero. The coefficient that multiplies the mode 2 coupling is independent of the aspect ratio of the surface.

The new results have been used to calculate the minimum possible Q_r for electrically small antennas contained within a prolate spheroid shape versus aspect ratio. Based on these results, a formula has been developed that gives the minimum Q_r for aspect ratios from 1 to 100. This formula has been used to determine a lower bound on Q_r for electrically small antennas contained within a cylindrical shape. For large aspect ratios of the cylinder, the lower bound gives a much larger value for the minimum Q_r than Chu's result. As the cylinder aspect ratio approaches 1, the cylinder bound and the Chu limit converge. The new bound is consistent with Q_r calculated for practical antennas.

CONTENTS

Figures

Tables

1. INTRODUCTION

In SPAWAR Systems Center San Diego (SSC San Diego) Technical Document (TD) 3188 (Adams and Hansen, 2004), we presented results for the quality factor of an antenna in prolate spheroidal coordinates for the lowest mode. The quality factor or "Q" was evaluated as a function of the aspect ratio (ratio of lengths of vertical to horizontal of an ellipsoid of revolution) for the lowest mode. An analytic formula was derived in the very long wavelength limit for Q as a function of the aspect ratio. This formula reduced to the famous one first derived by Chu (1947) for a sphere. Adams and Hansen (2005) extended these results to oblate spheroids. Figures 1 and 2 present the major calculation results. For all cases, the polarization is vertical. We will also assume that all quantities are independent of the azimuth angle, ϕ.

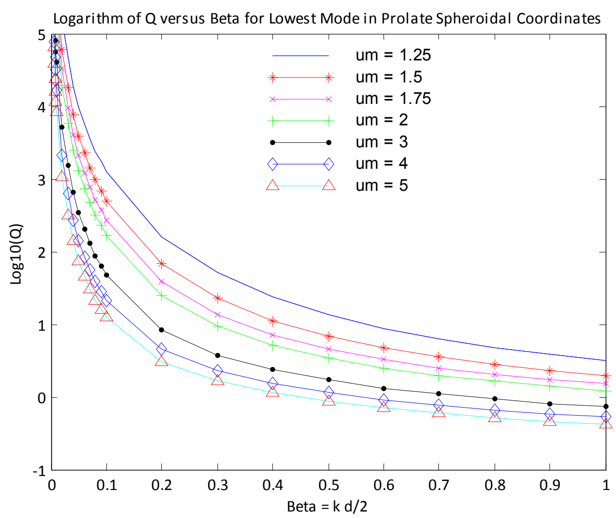


Figure 1. Value of logarithm of Q versus β for the lowest mode in prolate spheroidal coordinates for seven values of the aspect ratio.

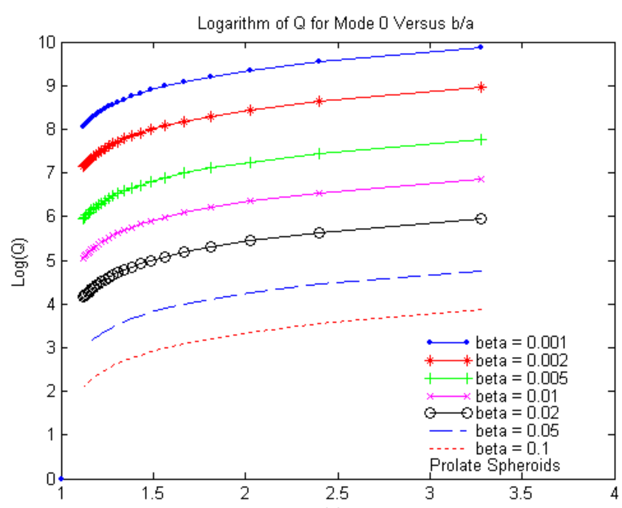


Figure 2. Value of logarithm of Q versus aspect ratio computed for the lowest mode in prolate spheroidal coordinates for seven values of β.

The distance between foci of the spheroidal surface is defined as d. The dimensionless wave number β is equal to the ratio of π d to the wavelength λ. The minimum value of u (u_m) is related to the aspect ratio of the surface that encloses the antenna. The semi-major axis is b and the semi-minor axis is defined as a. The relationship between u_m and b/a is given by

$$b/a = u_m/\sqrt{(u_m^2-1)}. \tag{1a}$$

As the value of u_m approaches 1, the ratio of the height to width of the enclosing surface approaches infinity. The value for the sphere is obtained as u_m approaches infinity. Many of the formulas make a transition to spherical coordinates as u_m becomes large and d approaches 0, such that the product $d \cdot u_m$ approaches the diameter of a sphere. For a prolate ellipsoid, b is greater than a. For an oblate ellipsoid, a is greater than b. For both prolate and oblate spheroidal coordinates, the height, b, is related to d and u_m by

$$b = \tfrac{1}{2}\, d\, u_m. \tag{1b}$$

For prolate spheroidal surfaces, the width, a, is given by

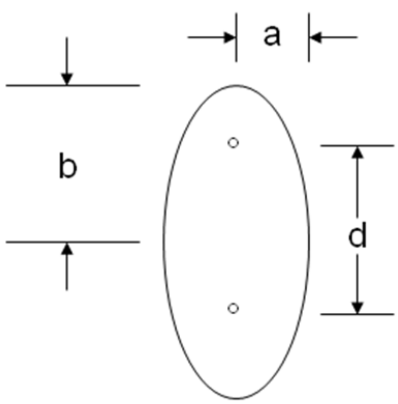
$$a = \tfrac{1}{2}\, d \sqrt{(u_m^2-1)}. \tag{1c}$$

In prolate spheroidal coordinates, the minimum value of u_m is 1. The vertically oriented body confined to such a surface is long and thin. For oblate spheroidal surfaces the width is given by

$$a = \tfrac{1}{2}\, d \sqrt{(u_m^2+1)}. \tag{1d}$$

For oblate spheroidal coordinates, the minimum value of u_m is 0. This case corresponds to a disk in the horizontal plane.

Figure 3 describes the surface geometry. Independent variables η, u, and ϕ are defined in terms of x, y, and z. The parameters of the surface, d, a, b, and u_m, are also defined.



Prolate:

$X= (d/2) \sqrt{(u^2-1)} \sqrt{(1-\eta^2)} \cos(\varphi)$

$Y=(d/2) \sqrt{(u^2-1)} \sqrt{(1-\eta^2)} \sin(\varphi)$

$Z=(d/2)\, u\, \eta$

$u_m = b/a/ \sqrt{((b/a)^2-1)}$

Oblate:

$X=(d/2) \sqrt{(u^2+1)} \sqrt{(1-\eta^2)} \cos(\varphi)$

$Y=(d/2) \sqrt{(u^2+1)} \sqrt{(1-\eta^2)} \sin(\varphi)$

$Z=(d/2)\, u\, \eta$

$u_m = b/a/ \sqrt{(1-(b/a)^2)}$

Figure 3. Description of geometry of surface and definition of variables.

The analytic formulas for Q for mode 0 as a function of β and u_m on the long wavelength limit are given by

$$Q_p = (3/4\beta^3)\ \{u_m/\ (u_m^2-1) - 0.5u_m^{-1} + 0.25\ (u_m^{-2}-1)\ \ln((u_m+1)/\ (u_m-1))\} \qquad (2a)$$

for prolate spheroidal coordinates, and

$$Q_o = (3/4\beta^3)\ \{-u_m/(u_m^2+1) + 0.5u_m^{-1} + 0.5\ (u_m^{-2}+1)\ \tan^{-1}(1/u_m)\} \qquad (2b)$$

for oblate spheroidal coordinates. Both cases merge into Chu's result:

$$Q_s = (2\pi\ a_s/\lambda)^{-3} \qquad (2c)$$

for a sphere of radius a_s.

Having analytic formulas allows us to vary easily the height or aspect ratio while keeping all other parameters constant. In the graphs above, we vary the dimensionless wave number for six aspect ratios, or we keep the wave number constant (very small) while varying the height or volume.

In the results above, only the lowest mode was considered. Modes are defined by solutions of the angular eigenfunction equation (to be described shortly). The lowest mode has no nodes, as the polar coordinate θ varies from -90 to 90º. The first mode has a null at $\theta = 0°$, the second node has two nulls, and the eigenvalues increase monotonically. For any physically realizable current, one expects a combination of modes to be excited.

As explained in SSC San Diego TD 3188, having a low value of Q is important for electrically small antennas. We seek to find the combination of modes that minimizes Q. We will concentrate on prolate spheroidal coordinates. The oblate formulas can be obtained by the transformation $u_p \rightarrow j\ u_o$ and $d_p \rightarrow -j\ d_o$. The parameter β is directly related to d and changes accordingly.

2. MODAL DECOMPOSITION

A solution for the source-free electromagnetic fields is a sum of the product of the functions:

$$e(\beta, u, \eta, t) = \exp(2\, j\, \beta\, c\, t/d) \sum w^{(n)} (\beta, u_m)\, S^{(n)}(\beta, \eta)\, R^{(n)}(\beta, u), \tag{3}$$

where $S^{(n)}(\beta, \eta)$ is a solution to the angular eigenvalue equation ($\eta = \cos\theta$) for prolate ellipsoids (Abramowitz and Stegun, 1965),

$$0 = ((1-\eta^2)S^{(n)}{}_\eta)_\eta + (\kappa^{(n)} - \beta^2\, \eta^2)S^{(n)}, \tag{4a}$$

and $R_n(\beta, u)$ is a solution for the radial equation,

$$0 = ((u^2-1)R^{(n)}{}_u)_u + (\beta^2\, u^2 - \kappa^{(n)})R^{(n)}. \tag{4b}$$

The parameter $\kappa^{(n)}$ is the eigenvalue that results from solving Equation (4a), subject to the boundary condition that the solution must be finite for all $-1 \le \eta \le 1$. Mode description was provided in the previous section.

In general, the eigenvalues and eigenfunctions depend on β. For $\beta = 0$, these parameters become their counterparts to Legendre and Bessel functions. In the expressions given above and in subsequent ones, a subscript denotes a derivative with respect to the variable and a superscript with parentheses denotes a mode number. The subscript rule has a two exceptions, the fundamental constants μ_o and ε_o. A superscript asterisk denotes the complex conjugate. In the volume exterior to the volume, the radical function $R^{(n)}$ will describe outwardly propagating waves.

The expansion coefficients $w^{(n)}$ generally depend on the wavenumber β and the parameter characterizing the surface u_m. Although the expansion coefficients can be complex, we will restrict the solution space for finding the minimum Q to real values for these quantities. Some quantities must be real and positive definite.

For a vertically polarized electromagnetic field, the electric field is in the $\underline{\mathbf{u}}$ and $\underline{\boldsymbol{\eta}}$ direction (we denote a unit vector by a bold symbol that is underlined. A vector will be printed as a bold capital letter). The magnetic field is in the $\underline{\boldsymbol{\varphi}}$ direction. In terms of a function, $e(\beta, u, \eta)$, the expressions for the two components of the electric field and one component of a magnetic field are given by

$$\begin{aligned}\mathbf{E} = {} & \eta\, \underline{\mathbf{u}}\, \{\beta^2 e + ((e(u^2-1))_u/(u^2-\eta^2))_u + (u\, (e(1-\eta^2))_\eta/(u^2-\eta^2))_u /\eta\}(4/d^2)\sqrt{(u^2-1)}/(u^2-\eta^2) \\ & + u\, \underline{\boldsymbol{\eta}}\, \{\beta^2 e + (\eta(e(u^2-1))_u/(u^2-\eta^2))_\eta /u + ((e(1-\eta^2))_\eta/(u^2-\eta^2))_\eta \}(4/d^2)\sqrt{(1-\eta^2)}/(u^2-\eta^2)\end{aligned} \tag{5a}$$

$$\mathbf{B} = -4\, j\, \beta\, \underline{\boldsymbol{\varphi}}\, (u e_u - \eta e_\eta) \bullet \{c d^2(u^2-\eta^2)\}^{-1}\sqrt{(u^2-1)}\, \sqrt{(1-\eta^2)} \tag{5b}$$

The expression for Q is the ratio of the energy per unit time stored in the total volume to the power radiated. The expression is given by:

$$Q = 2\, \omega\, W_{stored} / P\, (u\rightarrow\infty), \tag{6}$$

where the stored energy is given by

$$W_{stored} = 2\pi\, (d/2)^3 \int_{um}^{\infty} \int_{-1}^{1} (u^2-\eta^2)\, du\, d\eta\, (\varepsilon_o\, \mathbf{E}\cdot\mathbf{E}^*/4 + \mathbf{B}\cdot\mathbf{B}^*/(4\mu_o) - \underline{\mathbf{u}}\cdot\mathbf{E} \times \mathbf{B}^*/(2\mu_o\, c)). \tag{7}$$

The expression above is motivated by the usual expression for the energy in the electromagnetic field. We seek to calculate the stored electromagnetic energy external to a surface of the ellipsoid of revolution. The lower limit on the integral over u restricts the calculation

to the volume external to the surface. The first two terms are the usual expression for the total electromagnetic energy per unit volume. Subtracting the third term, the energy radiated to infinity results in calculating the energy stored in the field. Without subtracting that third term, the integral would be infinite. This expression was adapted from one in the paper by McLean (1996). The radiated energy is given by

$$P\,(u\to\infty) = 2\pi\,(d\,u/2)^2 \int_{-1}^{1} \underline{\mathbf{u}}\cdot(\mathbf{E} \times \mathbf{B}^*)/(2\mu_o)\, d\eta. \tag{8}$$

The expression for the radial eigenfunctions, $R^{(n)}$ in terms of the angular eigenfunctions, is given by Flammer (1957, pp. 53–54):

$$R^{(n)}(\beta, u) = -j^{(n+1)} /[2\,\beta\, d^{(n)}(\beta)] \int_{-1}^{1} d\eta\, (u^2+\eta^2-1)^{-1/2} \exp[j\,\beta\,(u^2+\eta^2-1)^{1/2}]\, S^{(n)}(\beta, \eta) \tag{9a}$$

for n even, and

$$R^{(n)}(\beta, u) = -j^{n-1}\, 3u /[2\, d^{(n)}(\beta)] \int_{-1}^{1} d\eta\, \eta\, (u^2+\eta^2-1)^{-1} \{1+j[\beta(u^2+\eta^2-1)^{-1/2}]\} \exp[j\,\beta\,(u^2+\eta^2-1)^{1/2}]\, S^{(n)}(\beta, \eta). \tag{9b}$$

for n odd. The parameter $d^{(n)}$ is a normalization constant designed to ensure that as the radial coordinates become large, the radial functions become

$$R^{(n)}\,(\beta, u\to\infty) = (\beta\, u)^{-1} \exp(j\,\beta\, u - (n-1)\,\pi / 2). \tag{9c}$$

The normalization for the radial eigenfunction leads to the definition of $d^{(n)}$ as

$$d^{(n)}(\beta) = 0.5 \int_{-1}^{1} d\eta\, S^{(n)}(\beta, \eta) \text{ for n even,} \tag{10a}$$

or

$$d^{(n)}(\beta) = 1.5 \int_{-1}^{1} d\eta\, \eta\, S^{(n)}(\beta, \eta) \text{ for n odd.} \tag{10b}$$

Using the modal decomposition of the basic function given in Equation (3), the final expression for the radiated power is given by

$$P\,(u\to\infty) = (4\pi\beta^2/d^2 c\mu_o) \sum_{n=0,}^{\infty} \sum_{l=0}^{\infty} \exp(j(l-n)\pi/2)\, w^{(n)}\, w^{(l)*} \int_{-1}^{1} d\eta\, (1-\eta^2)\, S^{(n)}(\beta, \eta) S^{(l)}(\beta, \eta). \tag{11}$$

If one mode is excited, the expression for the radiated power is guaranteed to be positive definite. There are physical reasons to presume that this expression must be positive for all realizable combinations of modes. In the limit of small electrical size (β small compared to 1), the eigenfunctions reduce to Legendre polynomials and the double summation reduces to a single sum with three terms.

$$\begin{aligned} P\,(\beta\to 0) = (4\pi\beta^2/d^2 c\mu_o) \sum_{n=0}^{\infty} w^{(n)}\, \{ & w^{(n)*}\, 4\,(n^2+n-1)/[(2n+1)(2n-1)(2n+3)] \\ & + 2\, w^{(n+2)*}\, (n+2)(n+1)/[(2n+1)(2n+3)(2n+5)] \\ & + 2n(n-1)\, w^{(n-2)*} /[(2n-3)(2n-1)(2n+3)]\}. \end{aligned} \tag{12}$$

The corresponding expression for the stored energy is given by

$$\begin{aligned} W_{stored} = (\pi\, \varepsilon_o / d) \sum_{n=0,}^{\infty} \sum_{l=0}^{\infty} w^{(n)}\, w^{(l)*} \int_{um} \int_{-1}^{1} (u^2-\eta^2)\, du\, d\eta\, [& \eta^2\, (u^2-1)\, (u^2-\eta^2)^{-3}\, x^{(n)}\, x^{(l)*} \\ & + u^2\, (1-\eta^2)\, (u^2-\eta^2)^{-3}\, y^{(n)}\, y^{(l)*} \\ & + \beta^2\, (u^2-1)\, (1-\eta^2)\, (u^2-\eta^2)^{-2}\, z^{(n)}\, z^{(l)*} - 2\beta\, u\, (u^2-1)^{1/2}\, (1-\eta^2)\, (u^2-\eta^2)^{-5/2}\, y^{(n)}\, z^{(l)*}. \end{aligned} \tag{13a}$$

The integrand of Equation (13a) has three distinct terms. In terms of the eigenfunctions, they are as follows:

$$x^{(n)} = (\kappa^{(n)} - \beta^2 \eta^2)R^{(n)} S^{(n)} - (1-\eta^2)(u^2+\eta^2) R^{(n)}S^{(n)}{}_{\eta} /[\eta(u^2-\eta^2)] + u(1-\eta^2) S^{(n)}{}_{\eta} R^{(n)}{}_u / \eta - 2u(u^2-1) S^{(n)} R^{(n)}{}_u/(u^2-\eta^2), \tag{13b}$$

$$y^{(n)} = (\beta^2 u^2 - \kappa^{(n)}) R^{(n)} S^{(n)} + 2\eta(1-\eta^2) R^{(n)} S^{(n)}{}_{\eta} /(u^2-\eta^2) + (u^2-1)(u^2+\eta^2)R^{(n)}{}_u S^{(n)}/ [u(u^2-\eta^2)] + \eta(u^2-1) R^{(n)}{}_u S^{(n)}{}_{\eta} /u, \tag{13c}$$

$$z^{(n)} = S^{(n)} u R^{(n)}{}_u - R^{(n)}\eta S^{(n)}{}_{\eta}. \tag{13d}$$

In general, the expressions in Equations (11) and (12) can be expressed as double sums of functions that must be computed numerically:

$$W_{stored} = (\pi \varepsilon_o /d) \sum \sum w^{(n)} w^{(l)*} \quad M^{(n,l)}, \tag{14a}$$

$$P(u\to\infty) = (4\pi\beta^2/d^2c\mu_o) \sum \sum \exp(j(l-n)\pi/2)w^{(n)} w^{(l)*} N^{(n,l)}, \tag{14b}$$

and

$$Q = \beta^{-1} \sum \sum w^{(s)} w^{(q)*} \quad M^{(s,q)} [\sum \sum \exp(j(l-n)\pi/2) w^{(n)} w^{(l)*} N^{(n,l)}]^{-1}. \tag{15}$$

What remains to be done is to compute the coefficients $M^{(s,q)}(\beta, u_m)$ and $N^{(n,l)}(\beta)$ and vary the weights of the modes $w^{(n)}$ until a minimum of Q is determined. We will divide the expression in Equation (15) by $w^{(0)}$, the weight of mode 0. Using the methods described in the appendix, we computed the eigenvalues $\kappa^{(n)}$, the eigenfunctions $S^{(n)}(\beta, \eta)$ and $R^{(n)}(\beta, u)$, and various normalization constants. Table A-1 presents a computation of the eigenvalues for six modes as a function of β. Table A-2 presents a computation of the normalization constant $d^{(n)}(\beta)$. Tables A-3 through A-5 present the eigenfunction, the normalization constant, and the derivative of the angular eigenfunction. Tables A-6 and A-7 present the corresponding for the radial eigenfunction and their derivatives for six modes.

The symmetry of the modes imposes a small restriction on the summation. The modes are even or odd with respect to reflection about the origin. The integrals in Equations (9) and (11) will equal zero unless n and l are both even or odd. The term $\exp(j(l-n)\pi/2)$ is ±1.

3. FIELD BEHAVIOR AT LARGE DISTANCES

The need for convergent integrals requires careful consideration about the form of energy. If the third term in the expression for the energy ($\underline{\mathbf{u}}\cdot\mathbf{E} \times \mathbf{B}/(2u_o c)$) were absent, the integral would diverge linearly for large values of u. The addition of that third term in Equation (7) eliminates the linearly divergent part. Thus, the remainder decreases at least as fast as u^{-1}. To prevent the integral from diverging logarithmically, this term must also be cancelled.

We will concentrate on the case of n even. The extension to the case of n odd is straight-forward. For large u, the exponential in the integrand of the radial eigenfunction in Equation (9a) is given by

$$\exp(j\,\beta\,\surd(u^2 + \eta^2 - 1) \sim \exp(j\,\beta\,u)\,[1 + j\,\beta\,(\eta^2 - 1)/(2u)]. \tag{16}$$

In this limit, the radial eigenfunction is given by:

$$R^{(n)}(\beta, u) = -j^{(n+1)}\,(\beta u)_{-1}\,[1 - j\beta/(2u) + j\beta\,(2u)^{-1} \int_{-1}^{1} d\eta\eta^2 S^{(n)}(\beta, \eta)/\int_{-1}^{1} d\eta\, S^{(n)}(\beta, \eta)]. \tag{17}$$

It is convenient to define a parameter $A^{(n)}$ by the expression,

$$A^{(n)} = \int_{-1}^{1} d\eta\eta^2 S^{(n)}(\beta,\eta)/\int_{-1}^{1} d\eta\, S^{(n)}(\beta, \eta). \tag{18}$$

The derivative with respect to u is given by

$$R^{(n)}{}_u(\beta, u) = j^n\,(\beta u)_{-1}\,[1 - j\beta/(2u) + j(\beta u)^{-1} + j\beta\,(2u)^{-1} A^{(n)}]. \tag{19}$$

A straightforward evaluation of the terms for the modal decomposition of **E** and **B** is given by

$$\mathbf{E}^{(n)} \cdot \mathbf{E}^{(m)} = 16\,(1-\eta^2)\,(-1)^{(n+m)/2}\,[\beta^2 + j\,\beta^3\,(2u)^{-1}\,(A^{(n)} - A^{(m)})]\,S^{(n)}\,S^{(m)}\,(d^4\,u^2)^{-1} \tag{20}$$

and

$$\mathbf{B}^{(n)} \cdot \mathbf{B}^{(m)} = 16\,\beta^2\,(1-\eta^2)\,(-1)^{(n+m)/2}\,[S^{(n)}\,S^{(m)} + j\beta\,S^{(n)}\,S^{(m)}\,(A^{(n)} - A^{(m)})/(2u) + j\,\eta\,(S^{(m)}\,S_\eta{}^{(n)} - S^{(n)}\,S_\eta{}^{(m)})/(\beta u)]\,(d^2\,c\,u^2)^{-1}. \tag{21}$$

The term that renders the stored energy to be finite is then given by

$$\underline{\mathbf{u}}\cdot\mathbf{E}^{(n)} \times \mathbf{B}^{(m)} = 16\,\beta\,(-1)^{(n+m)/2}\,(1 - \eta^2)[\beta\,S^{(n)}S^{(m)} - j\beta^2\,S^{(n)}\,S^{(m)}\,(A^{(m)} - A^{(n)})/(2u) - j\,S^{(n)}\,\eta\,S_\eta{}^{(m)}/u]\,(d^4\,c\,u^2). \tag{22}$$

Thus, the integrand for the stored energy is given by

$$\varepsilon_o\,\mathbf{E}^{(n)}\cdot\mathbf{E}^{(m)}/4 + \mathbf{B}^{(n)}\cdot\mathbf{B}^{(m)}/(4\,\mu_o) - \underline{\mathbf{u}}\cdot\mathbf{E}^{(n)} \times \mathbf{B}^{(m)}/(2\,\mu_o\,c) = 16\,\beta\,(-1)^{(n+m)/2}\,(1-\eta^2)\,[-j\beta^2\,S^{(n)}\,S^{(m)}\,(A^{(m)} - A^{(n)})/(2u) + j\,(S^{(m)}\,\eta\,S_\eta{}^{(n)} - S^{(n)}\,\eta\,S_\eta{}^{(m)})/u]\,(d^4\,c\,u^2)^{-1}. \tag{23}$$

The primary feature to note is that the term proportional to u^{-3} is anti-symmetric in the indices m and n. If the coefficients $w^{(n)}$ are real, these terms will cancel (the term m and n will cancel with n and m). Unfortunately when dealing with an individual term, the anti-symmetry is not evident. If the terms are symmetrized, the accuracy is maintained and the integrals converge absolutely. This was the adopted procedure in computing the integrals.

4. THE APPROXIMATION OF VERY SMALL ELECTRICAL SIZE

Simplifications to the computation occur for the important case of small electrical size. In dealing with mode 0, the simplifications were dramatic. Terms such as a derivative of the eigenfunction with respect to the coordinate η could be dropped because it was of order β^2. The eigenvalue $\kappa^{(0)}$ was also of order β^2. For the general case of a higher mode, the simplifications are significant, but not as dramatic. The eigenvalue for the mode n is given by

$$\kappa^{(n)} = n(n+1) + \beta^2 (2n^2+2n-1)/[(2n-1)(2n+3)]. \quad (24)$$

The eigenfunction is equal to the Legendre polynomial $P^{(n)}(\eta)$, with corrections of order β^2. In computing the radial eigenfunctions, this simplification results in analytic formulas, as long as the product βu is small. In this case for n even, the radial eigenfunction is given by

$$R^{(n)} = (-1)^n j (2\beta d^{(n)})^{-1} \int_{-1}^{1} d\eta\, P^{(n)}(\eta) (u^2+\eta^2-1)^{-1/2}. \quad (25)$$

The expressions for the radial eigenfunctions for modes 0 and 2 are given by Dwight (1961, p. 59):

$$R^{(0)} = -j (2\beta d^{(0)})^{-1} \ln[(u+1)/(u-1)] \quad (26a)$$

and

$$R^{(2)} = j (2\beta d^{(2)})^{-1} [3u - ¼(3u^2-1) \bullet \ln\{(u+1)/(u-1)\}]. \quad (26b)$$

In the small β limit, the constants $d^{(0)}$ and $d^{(2)}$ are given by 1 and $\beta^2/45$, respectively. Thus, in this limit, the imaginary part of the radial eigenfunction $R^{(0)}$ is proportional to β^{-1} and $R^{(2)}$ is proportional to β^{-3}. In general, the n-th order eigenfunctions are proportional to $\beta^{-(n+1)}$, which will cause the functions multiplying the mode expansion coefficients to be proportional to powers of β. We computed the integrals, $M^{(l,\, m)}$, $N^{(l,\, m)}$, in the mode expansion for modes 0,0, 0,2, 2,2, 0,4, 2,4, and 4,4. The results for very small β are as follows:

$$M^{(0,0)} = m^{(0,0)} \beta^{-2}, \quad (27a)$$

$$M^{(0,2)} = m^{(0,2)} \beta^{-4}, \quad (27b)$$

$$M^{(2,2)} = m^{(2,2)} \beta^{-6}, \quad (27c)$$

$$M^{(0,4)} = m^{(0,4)}\beta^{-6}, \quad (27d)$$

$$M^{(2,4)} = m^{(2,4)}\beta^{-8}, \quad (27e)$$

$$M^{(4,4)} = m^{(4,4)}\beta^{-10}. \quad (27f)$$

We call the lowercase m's reduced coefficients. Adams and Hansen derived an analytic formula for $m^{(0,0)}$ for very small values of β. This formula is

$$m^{(0,0)} = u_m/(u_m^2-1) - ½\, u_m^{-1} + ¼\, (u_m^{-2}-1) \ln[(u_m+1)/(u_m-1)]. \quad (28)$$

The accuracy of the formula involves neglecting the next higher power in β. Table 1 was numerically computed by ensuring that the integrals evaluated at $\beta = 0.001$ and $\beta = 0.01$ varied by the appropriate power of 10. In general, the reduced coefficients $m^{(n,l)}$ are functions only of the aspect ratio through their dependence on u_m. The table presents the values of these coefficients as a function of u_m.

Table 1. Reduced coefficients for modes 0, 2, and 4 versus u_m.

$m^{(0,0)}$	$m^{(0,2)}$	$m^{(2,2)}$	$m^{(0,4)}$	$m^{(2,4)}$	$m^{(4,4)}$	u_m
9.6834E+00	1.0766E+02	2.4224E+03	7.5911E+03	1.7079E+05	3.0005E+07	1.05
2.9688E+00	1.5139E+01	3.4062E+02	4.5314E+02	1.0195E+04	1.7913E+06	1.15
1.6245E+00	4.7021E+00	1.0580E+02	7.7749E+01	1.7493E+03	3.0735E+05	1.25
1.0561E+00	1.9243E+00	4.3297E+01	1.9761E+01	4.4461E+02	7.8117E+04	1.35
7.4822E-01	9.1630E-01	2.0617E+01	6.2749E+00	1.4119E+02	2.4806E+04	1.45
5.5873E-01	4.8179E-01	1.0840E+01	2.3117E+00	5.2013E+01	9.1384E+03	1.55
4.3259E-01	2.7209E-01	6.1220E+00	9.4907E-01	2.1354E+01	3.7518E+03	1.65
3.4402E-01	1.6230E-01	3.6517E+00	4.2361E-01	9.5310E+00	1.6746E+03	1.75
2.7932E-01	1.0112E-01	2.2752E+00	2.0218E-01	4.5490E+00	7.9925E+02	1.85
2.3062E-01	6.5301E-02	1.4693E+00	1.0199E-01	2.2947E+00	4.0317E+02	1.95
1.9307E-01	4.3459E-02	9.7782E-01	5.3901E-02	1.2128E+00	2.1308E+02	2.05
1.6354E-01	2.9677E-02	6.6773E-01	2.9650E-02	6.6710E-01	1.1721E+02	2.15
1.3993E-01	2.0723E-02	4.6628E-01	1.6886E-02	3.7994E-01	6.6754E+01	2.25
1.2079E-01	1.4758E-02	3.3205E-01	9.9160E-03	2.2310E-01	3.9199E+01	2.35
1.0507E-01	1.0694E-02	2.4062E-01	5.9830E-03	1.3461E-01	2.3652E+01	2.45
9.2036E-02	7.8709E-03	1.7709E-01	3.6987E-03	8.3219E-02	1.4621E+01	2.55
8.1115E-02	5.8746E-03	1.3218E-01	2.3372E-03	5.2585E-02	9.2391E+00	2.65
7.1890E-02	4.4407E-03	9.9915E-02	1.5064E-03	3.3894E-02	5.9551E+00	2.75
6.4038E-02	3.3958E-03	7.6405E-02	9.8872E-04	2.2245E-02	3.9084E+00	2.85
5.7309E-02	2.6244E-03	5.9049E-02	6.5976E-04	1.4844E-02	2.6081E+00	2.95
5.1505E-02	2.0481E-03	4.6082E-02	4.4702E-04	1.0057E-02	1.7671E+00	3.05
4.6470E-02	1.6128E-03	3.6287E-02	3.0717E-04	6.9109E-03	1.2143E+00	3.15
4.2080E-02	1.2806E-03	2.8813E-02	2.1385E-04	4.8112E-03	8.4533E-01	3.25

The coefficients in the denominator $N^{(n,l)}$ in the small β limit are given by finite constants if the difference between n and l is ≤2. For small β, these constants are given by

$$N^{(0,0)} = 4/3, \tag{29a}$$

$$N^{(0,2)} = -4/15, \tag{29b}$$

$$N^{(2,2)} = 4/21, \tag{29c}$$

$$N^{(2,4)} = -8/105, \tag{29d}$$

$$N^{(4,4)} = 76/693. \tag{29e}$$

The coefficient $N^{(0,4)}$ is of order β^2.

5. RESULTS FOR Q WHEN INCLUDING MODES 0 AND 2

Table 2 presents the results for $M^{(0,0)}$, $M^{(0,2)}$, and $M^{(2,2)}$ versus β for the value of $u_m = 1.25$. The scaling of the coefficients as a function of β is evident by comparing the values at β = 0.001 and β = 0.01. As β approaches 1, the scaling becomes less accurate. By β = 0.1, the scaling is accurate to 0.1%. If β = 1.0, the simple scaling is very inaccurate.

Table 2. Coefficients for modes 0 and 2 in modal expansion of Q.

β	$M^{(0,0)}$	$M^{(0,2)}$	$M^{(2,2)}$	$N^{(0,0)}$	$N^{(0,2)}$	$N^{(2,2)}$
0.001	1.6245E+06	4.7021E+12	1.0580E+20	1.3333E+00	-2.6667E-01	1.9048E-01
0.002	4.0612E+05	2.9388E+11	1.6531E+18	1.3333E+00	-2.6667E-01	1.9048E-01
0.003	1.8050E+05	5.8051E+10	1.4513E+17	1.3333E+00	-2.6667E-01	1.9048E-01
0.004	1.0153E+05	1.8368E+10	2.5829E+16	1.3333E+00	-2.6667E-01	1.9048E-01
0.005	6.4981E+04	7.5234E+09	6.7711E+15	1.3333E+00	-2.6667E-01	1.9048E-01
0.006	4.5127E+04	3.6282E+09	2.2676E+15	1.3333E+00	-2.6667E-01	1.9048E-01
0.007	3.3155E+04	1.9584E+09	8.9927E+14	1.3333E+00	-2.6667E-01	1.9048E-01
0.008	2.5385E+04	1.1480E+09	4.0359E+14	1.3333E+00	-2.6667E-01	1.9048E-01
0.009	2.0058E+04	7.1668E+08	1.9908E+14	1.3333E+00	-2.6667E-01	1.9048E-01
0.010	1.6247E+04	4.7021E+08	1.0580E+14	1.3333E+00	-2.6667E-01	1.9048E-01
0.020	4.0636E+03	2.9389E+07	1.6533E+12	1.3333E+00	-2.6666E-01	1.9048E-01
0.030	1.8074E+03	5.8055E+06	1.4516E+11	1.3333E+00	-2.6665E-01	1.9048E-01
0.040	1.0177E+03	1.8370E+06	2.5842E+10	1.3332E+00	-2.6664E-01	1.9048E-01
0.050	6.5223E+02	7.5251E+05	6.7760E+09	1.3331E+00	-2.6663E-01	1.9048E-01
0.060	4.5368E+02	3.6294E+05	2.2700E+09	1.3330E+00	-2.6661E-01	1.9048E-01
0.070	3.3397E+02	1.9593E+05	9.0056E+08	1.3329E+00	-2.6659E-01	1.9049E-01
0.080	2.5626E+02	1.1487E+05	4.0434E+08	1.3328E+00	-2.6656E-01	1.9049E-01
0.090	2.0299E+02	7.1722E+04	1.9955E+08	1.3326E+00	-2.6653E-01	1.9049E-01
0.100	1.6489E+02	4.7066E+04	1.0611E+08	1.3324E+00	-2.6650E-01	1.9049E-01
0.200	4.3047E+01	2.9503E+03	1.6727E+06	1.3298E+00	-2.6601E-01	1.9055E-01
0.300	2.0476E+01	5.8594E+02	1.4904E+05	1.3254E+00	-2.6520E-01	1.9064E-01
0.400	1.2570E+01	1.8692E+02	2.7085E+04	1.3193E+00	-2.6407E-01	1.9078E-01
0.500	8.9015E+00	7.7493E+01	7.2955E+03	1.3117E+00	-2.6263E-01	1.9096E-01
0.600	6.9005E+00	3.8013E+01	2.5269E+03	1.3025E+00	-2.6088E-01	1.9119E-01
0.700	5.6854E+00	2.1014E+01	1.0436E+03	1.2920E+00	-2.5882E-01	1.9148E-01
0.800	4.8899E+00	1.2700E+01	4.9135E+02	1.2800E+00	-2.5650E-01	1.9183E-01
0.900	4.3367E+00	8.2653E+00	2.5628E+02	1.2670E+00	-2.5386E-01	1.9225E-01
1.000	3.9349E+00	5.7050E+00	1.4525E+02	1.2528E+00	-2.5101E-01	1.9275E-01

If only modes 0 and 2 are included, the non-linear expression for Q versus the mode coefficients can be minimized without approximation. The expression for Q is given by

$$Q = \beta^{-1}\,[M^{(0,0)} + 2\,(w^{(2)}/w^{(0)})\,M^{(0,2)} + (w^{(2)}/w^{(0)})^2\,M^{(2,2)}]\,[N^{(0,0)} - 2\,(w^{(2)}/w^{(0)})\,N^{(0,2)} + (w^{(2)}/w^{(0)})^2\,N^{(2,2)}]^{-1}. \tag{30}$$

We minimize Q with respect to the ratio $w^{(2)}/w^{(0)}$ by differentiating and setting the derivative equal to 0. Normally, a cubic equation would result. In this case, the coefficient of the cubic term is zero. The result is a quadratic equation. The value for $w^{(2)}/w^{(0)}$ that minimizes Q is given by

$$w^{(2)}/w^{(0)} = [M^{(0,0)}N^{(2,2)} - M^{(2,2)}\,N^{(0,0)} + \sqrt{\{(M^{(2,2)}N^{(0,0)}-M^{(0,0)}N^{(2,2)})^2-4(M^{(0,2)}N^{(0,0)}-M^{(0,0)}N^{(0,2)})(M^{(2,2)}N^{(0,2)}-M^{(0,2)}N^{(2,2)})\}}][2(M^{(2,2)}N^{(0,2)}-M^{(0,2)}N^{(2,2)})]^{-1}. \tag{31}$$

An approximate solution that is valid for $\beta < 0.1$ is to neglect the variation of the denominator with respect to the coefficients $w^{(2)}/w^{(0)}$. For that case, the solution is given by

$$w^{(2)}/w^{(0)} = -M^{(0,2)}/M^{(2,2)}. \tag{32}$$

Using the scaling laws given in the equations results in $w^{(2)}/w^{(0)}$ being proportional to β^2. the result is

$$w^{(2)}/w^{(0)} = -(4/90)\beta^2. \tag{33}$$

The interesting feature is that this ratio is independent of u_m.

The ratio of mode 2 to mode 0 that minimizes the Q is $-4\beta^2/90$. In terms of quantities $m^{(n,l)}$ that are independent of β, the value of Q is given by

$$Q(\text{mode } 0, \text{small } \beta) = 0.75\ \beta^{-3}\ m^{(0,0)}. \tag{34}$$

The value of Q for modes 0 and 2 in which the contribution by the latter is chosen to minimize the value of Q is given by

$$Q(\text{modes } 0 \text{ and } 2) = 0.75\ \beta^{-3}\ m^{(0,0)}\ [1- m^{(0,2)}\ m^{(0,2)}/(m^{(0,0)}\ m^{(2,2)})]. \tag{35}$$

The dependence on β for modes 0 and 2 is a cubic one, the same as that for mode 0 alone. The ratio is independent of β. Table 1 lists the values of the $m^{(0,0)}$, $m^{(0,2)}$, and $m^{(2,2)}$. The reduction in Q in which the aspect ratio b/a is high can be substantial. If the aspect ratio is 3.3, the value of Q, by including mode 2, can be reduced by almost 50%.

Figure 4 presents the computed value of $z = m^{(0,2)} m^{(0,2)}/m^{(0,0)}\ m^{(0,2)}$ versus Log(b/a). The factor z represents the amount that inclusion of mode 2 reduces Q. As b/a approaches 1(u_m becomes large) the surface becomes more spherical, z decreases to 0. Thus, the result of TD 3188 for mode 0 dominates. This result, which reduces to that of Chu (1948) for a sphere, remains correct. When b/a becomes large (u_m decreasing to its minimum value of 1), z approaches 1. Thus, the inclusion of mode 2 decreases Q by large factors.

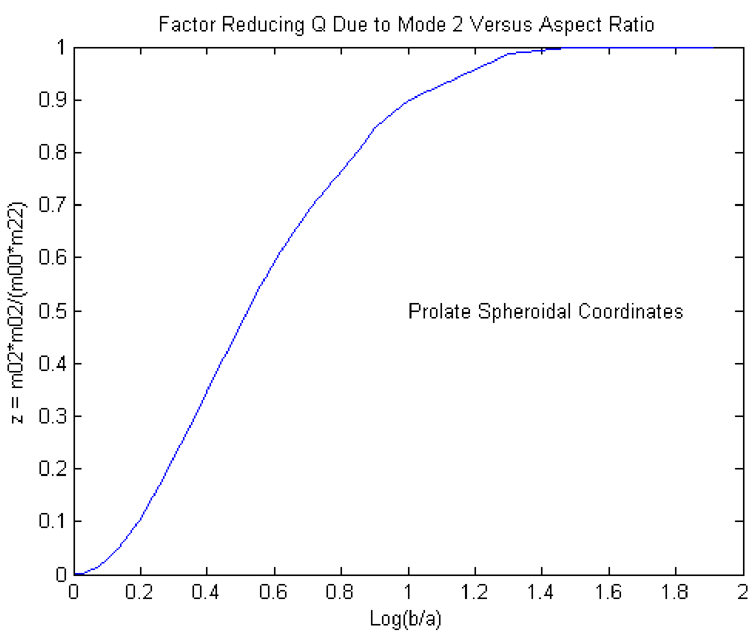


Figure 4. Value of modification of Q with inclusion of mode 2 versus b/a for small β.

6. RESULTS FOR Q WHEN INCLUDING MODES 0, 2, 4, AND 6

The scaling laws given in Equations (27a) through (27f) show that contributions for mode 2 should vary proportionally to β^2, and those for mode 4 should vary approximately according to β^4. For small β, the variation in the denominator of Q should be negligible. The minimization becomes a linear problem. Table 3 presents the coefficients $m^{(0,4)}$, $m^{(2,4)}$, and $m^{(4,4)}$ versus u_m. The variation in β for the terms that contribute to Q is provided in Equations (27a) through (27f).

Table 3. Reduced coefficients needed to consider contribution of mode 4 to the evaluation of Q for small β.

$m^{(0,4)}$	$m^{(2,4)}$	$m^{(4,4)}$	u_m
7.5911E+03	1.7079E+05	3.0005E+07	1.05
4.5314E+02	1.0195E+04	1.7913E+06	1.15
7.7749E+01	1.7493E+03	3.0735E+05	1.25
1.9761E+01	4.4461E+02	7.8117E+04	1.35
6.2749E+00	1.4119E+02	2.4806E+04	1.45
2.3117E+00	5.2013E+01	9.1384E+03	1.55
9.4907E-01	2.1354E+01	3.7518E+03	1.65
4.2361E-01	9.5310E+00	1.6746E+03	1.75
2.0218E-01	4.5490E+00	7.9925E+02	1.85
1.0199E-01	2.2947E+00	4.0317E+02	1.95
5.3901E-02	1.2128E+00	2.1308E+02	2.05
2.9650E-02	6.6710E-01	1.1721E+02	2.15
1.6886E-02	3.7994E-01	6.6754E+01	2.25
9.9160E-03	2.2310E-01	3.9199E+01	2.35
5.9830E-03	1.3461E-01	2.3652E+01	2.45
3.6987E-03	8.3219E-02	1.4621E+01	2.55
2.3372E-03	5.2585E-02	9.2391E+00	2.65
1.5064E-03	3.3894E-02	5.9551E+00	2.75
9.8872E-04	2.2245E-02	3.9084E+00	2.85
6.5976E-04	1.4844E-02	2.6081E+00	2.95
4.4702E-04	1.0057E-02	1.7671E+00	3.05
3.0717E-04	6.9109E-03	1.2143E+00	3.15
2.1385E-04	4.8112E-03	8.4533E-01	3.25

We define e and f by the relations:

$$e = \beta^{-2}\, w^{(2)}/w^{(0)} \tag{36a}$$

and

$$f = \beta^{-4}\, w^{(4)}/w^{(0)}. \tag{36b}$$

The expression for Q is then approximately

$$Q = 0.75\, \beta^{-3}\, [m^{(0,0)} + 2\, e\, m^{(0,2)} + e^2\, m^{(2,2)} + 2f\, m^{(0,4)} + 2\, e\, f\, m^{(2,4)} + f^2\, m^{(4,4)}]. \tag{37}$$

We minimize Q with respect to e and f, and the result for f is

$$f = [m^{(0,4)}\, m^{(2,2)} - m^{(0,2)}\, m^{(2,4)}]\, [m^{(2,4)}\, m^{(2,4)} - m^{(4,4)}\, m^{(2,2)}]^{-1}. \tag{38}$$

For all values of u_m, to five-place precision, the product of $m^{(0,4)}$ and $m^{(2,2)}$ equals the product of $m^{(0,2)}$ and $m^{(2,4)}$. Thus, f is zero! The value for e = -4/90 is independent of u_m. Something extraordinary is occurring to cause the minimum Q to occur with little contribution from mode 4. The inclusion of mode 2 significantly decreases the Q.

The extension to higher modes is straightforward as long as the variation due to β of the denominator in equation can be ignored. This should be valid for β less than 0.1. The minimization of Q with respect to n modes is then the solution to n linear algebraic equations. The higher modes are expected to contribute little for small β. The inclusion of mode 6 has been computed. The coefficient is very small, and its inclusion does not significantly change the mode 0, 2, and 4 results.

7. COMPARISON WITH A SPHERE

The Q for a spheroid from the previous section can be compared with that of a sphere in two important ways. The first way is to for equal heights. The second way is for equal volume. The first way is more typical.

In the cases described above, the Q is proportional to the third power of β, defined as the product of one-half the wavenumber and distance between foci of the ellipse. By equation,

$$d = 2b/u_m$$

so that

$$\beta^{-3} = (k\,b)^{-3}\,u_m^{\,3}. \tag{39}$$

The parameter b is the height of the spheroid. As the value of u_m becomes larger, the shape of the surface becomes more spherical and b approaches the radius of a sphere. To get the expression in the form in which the volume is equal, we use the expression for the volume of an ellipsoid of revolution,

$$V = 4\cdot\pi\cdot a^2\,b/3. \tag{40}$$

By equation, the relationship between a and d is given by

$$d = 2\cdot a\,\sqrt{(u_m^{\,2}-1)},$$

$$\beta^{-3} = (u_m^{\,2}-1)\cdot u_m\,4\cdot\pi\,/(3\cdot Vk^{-3}). \tag{41}$$

The above formula maintains equal volume.

Figure 5 presents the ratio of the Q for a prolate spheroid to that of a sphere with equal height. Figure 6 presents the corresponding case in which the volume is equal. Each graph also shows the result when considering only the lowest mode.

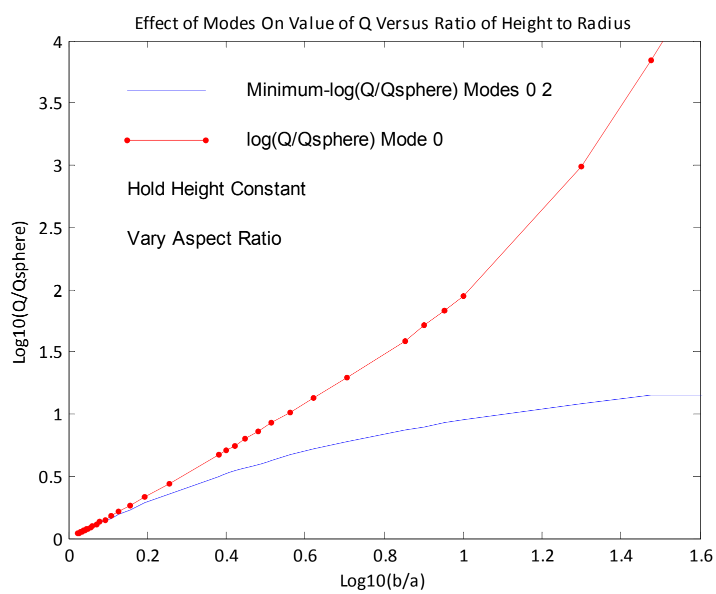


Figure 5. Logarithm of ratio of Q using only lowest mode to minimum Q using modes 0 and 2 versus logarithm of aspect ratio. Equal heights.

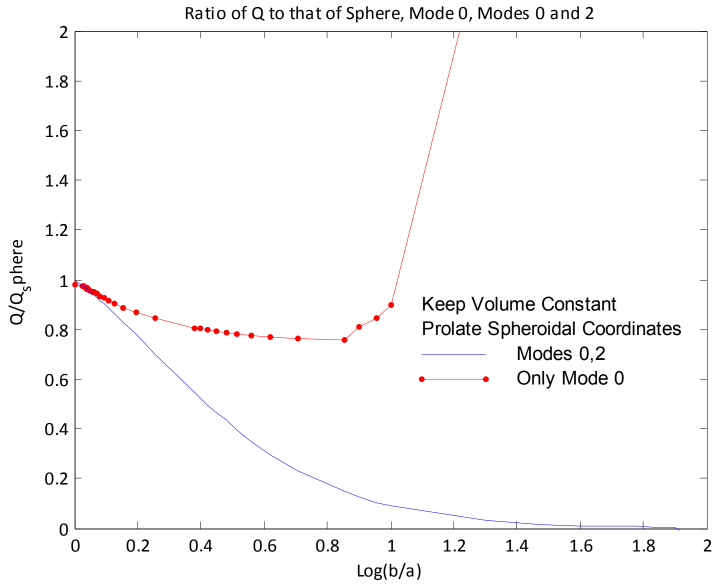


Figure 6. Logarithm of ratio of Q using only lowest mode to that of minimum using modes 0 and 2 versus aspect ratio. Equal volumes.

8. APPLICATIONS

8.1 PROLATE SPHEROID

Figure 5 shows the results of the calculations for the minimum Q_r for an antenna contained within a prolate spheroid shape. The results are presented as the ratio of the minimum Q_r for a prolate spheroid volume as a function of aspect ratio normalized to the minimum Q_r for a sphere having the same height. Also shown in the figure is the normalized minimum Q_r for the lowest order prolate spheroid mode. The Q_r for this mode is much larger than the true minimum when mode 2 is included.

At large aspect ratios, the Q_r with only the lowest mode included is larger than that of practical antennas (dipoles). This finding provided the clue that the minimum Q_r for prolate spheroid shapes required a sum of modes (Hansen and Adams, 2005). The minimum Q_r curve of Figure 5 was fit with an equation that gives an excellent fit for aspect ratios from 1 to 100 as follows:

$$\frac{\min Q_{ps}(0,2)}{\min Q_{sph}(r=b)} = A + B\cdot(b/a) + C\cdot\ln^2(b/a) + D\cdot\sqrt{b/a} + \frac{E}{\sqrt{b/a}}\,, \qquad (42)$$

where

min $Q_{ps}(0,2)$ is the minimum Q for the prolate spheroidal shape when modes 0 and 2 are included (our conjecture is that this is the minimum possible Q when all modes are included).

min $Q_{sph}(r = b)$ is the minimum Q for a sphere of radius r equal to the prolate spheroid semi-major axis (b).

a is the prolate spheroid semi-minor axis.

b is the prolate spheroid semi-major axis ($b \geq a$).

b/a is the aspect ratio of the prolate spheroid ($b/a \geq 1$).

The constants A, B, C, D, and E are as follows:

A	B	C	D	E
63.87347	1.08145	8.282164	-31.4191	-32.5363

Figure 7 shows a plot of the numerically calculated data using Equation (43), which is a repeat of Figure 5, compared with Equation (47) for values of aspect ratio up to 100. The curve labeled “data” is from Figure 5 (Minimum – Log(Q/Qsphere) Modes 0,2). The curve labeled “curvefit” is calculated using Equation (43). The curvefit data differ from the original numerically calculated data by less than 0.4%.

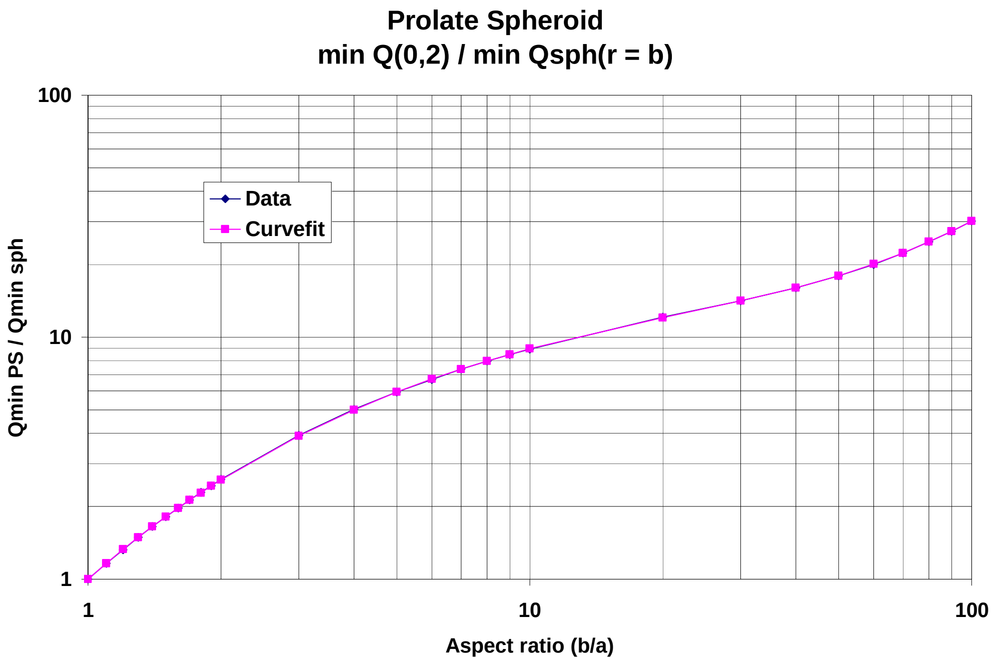


Figure 7. Minimum Q_r for electrically small prolate spheroid versus aspect ratio.

8.2 CYLINDER

Many practical antennas are more nearly fitted by a cylinder than by a prolate spheroid. To compare the performance of one of these antennas with our formula, we use a prolate spheroid with semi-major axis "b" and semi-minor axis "a" that just contains the cylinder of height "h" and radius "r" as illustrated in Figure 8. Note that in this case we are working with a perfect ground plane. By image theory, the radiation Q of half the antenna above a ground plane is the same as the full antenna in free space.

Figure 8 represents figures of revolution around the Z axis. The cylinder shown has a height of 10 and radius of 1, or an aspect ratio (h/r) of 10. The height would correspond to the half-length in free space. Correspondingly, the cylinder is fit with a half prolate spheroid. A continuum of half prolate spheroids just contain the cylinder. The figure shows three examples. For the prolate spheroid to fit tightly over the cylinder at the edge, the value of "a" must be greater than the radius of the cylinder (r > a). The value of "b" is then determined by the value of "a" and the height and radius of the cylinder (h, r) from the following formula:

$$b = \frac{h}{\sqrt{1-(r/a)^2}} \tag{43}$$

As illustrated in Figure 8, the prolate spheroid is taller when closer "a" is selected.

The volume of the half prolate spheroid varies, depending on the selection of "a". The formula for the volume of the half prolate spheroid just containing the cylinder is

$$Volume = \frac{1}{2}\cdot\frac{4}{3}\cdot\pi\cdot a^2\cdot b\,. \tag{44}$$

A minimum volume half prolate spheroid exists that just contains the cylinder. The parameters for the minimum volume half prolate spheroid can be derived by substituting Equation (42) into equation (43), and setting the derivative with respect to "a" equal to zero. This result is $a = \sqrt{(3/2)} \cdot r$, and it follows from Equation (47) that $b = \sqrt{3} \cdot h$ for the minimum volume half prolate spheroid that just contains the cylinder. Note that for this case, $b/a = \sqrt{2} \cdot (h/r)$.

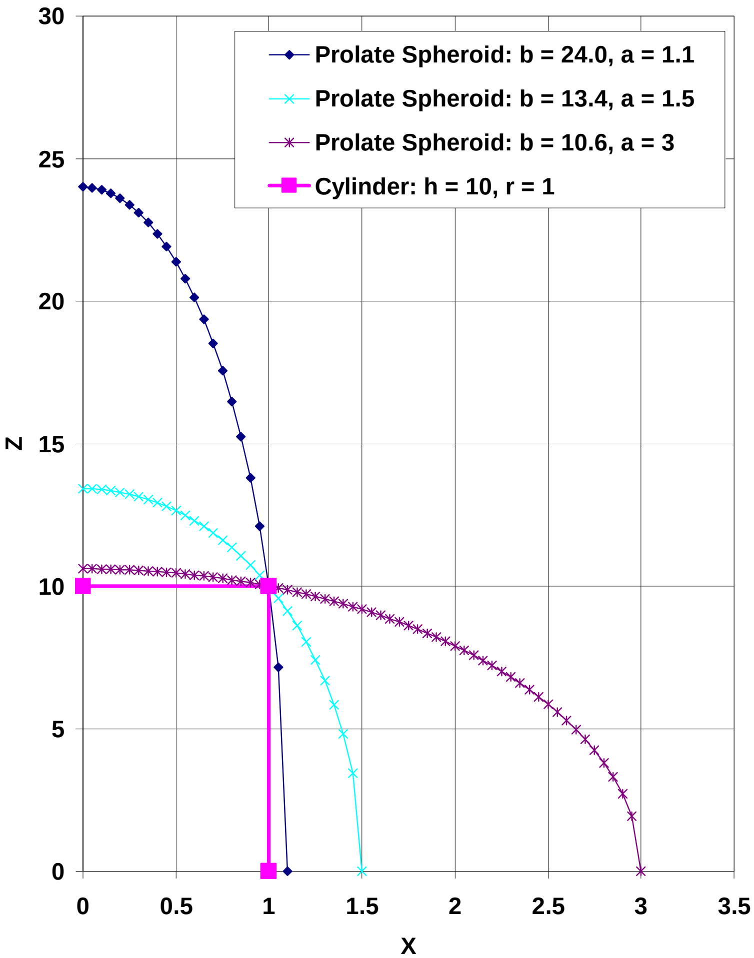


Figure 8. Half prolate spheroid containing cylinder.

Equations (42) and (43) have been used to plot the normalized minimum Q_r for a half prolate spheroid just containing a cylinder as a function of the prolate spheroid aspect ratio (b/a). An example for a cylinder having aspect ratio 10 is shown in Figure 9. Note that Q_r is normalized to the Chu minimum Q_r for a sphere having a radius equal to b, with $b \ll \lambda$.

The figure shows that maximum Q_r occurs when the prolate spheroid has an aspect ratio of 5.4. The volume of the half prolate spheroid that just contains the cylinder, normalized to the cylinder volume, is also plotted in Figure 9. Note that the maximum Q_r occurs at a different aspect ratio than the minimum volume. The minimum volume is considerably less than the volume at maximum Q_r.

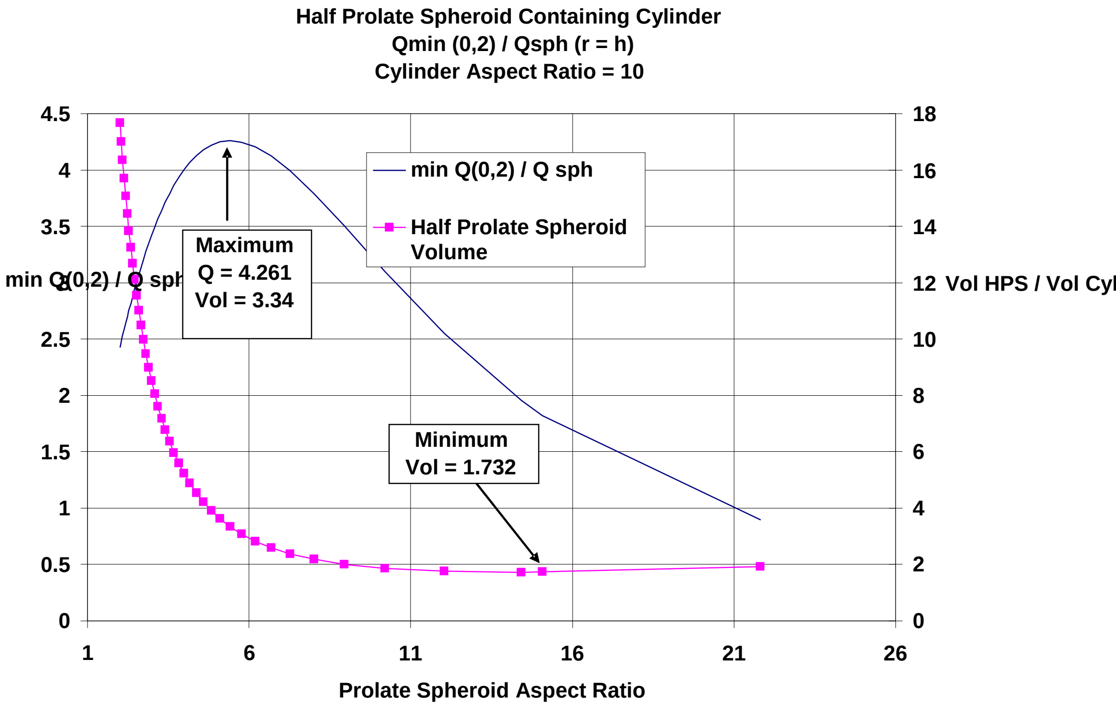


Figure 9. Minimum Q_r for half prolate spheroid fitting a cylinder of aspect ratio 10.

A second example for a cylinder having aspect ratio of 1.1 is shown in Figure 10. In this case, the location of the maximum Q_r and minimum volume are closer, but still not the same. The minimum volume is only a little less than the volume at maximum Q. The maximum Q_r is less than the previous case because the cylinder has more volume.

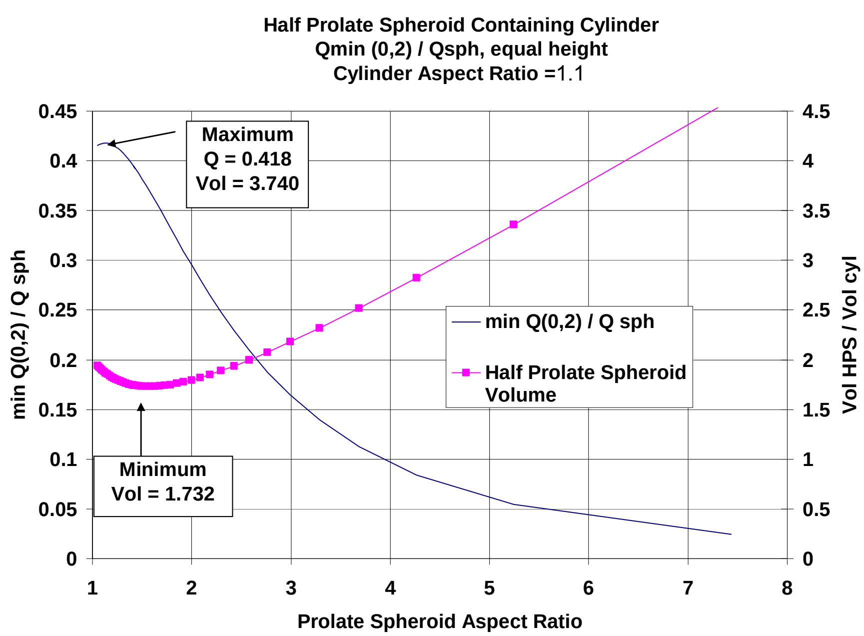


Figure 10. Minimum Q_r for half prolate spheroid fitting a cylinder of aspect ratio 1.1.

8.2.1 Cylinder Q_r Lower Bound

The objective was to develop a lower bound for the minimum Q_r of cylindrical shapes versus aspect ratio by using the results for the prolate spheroid. The concept is that the minimum Q_r for the cylindrical shape must be greater than the maximum of the minimum Q_r for all possible half prolate spheroids containing the cylinder. The rationale is that the cylinder has less height and less volume than any half prolate spheroid that contains it, and therefore must have a greater minimum Q_r.

The bound is calculated for each cylinder aspect ratio by taking the maximum of the minimum Q_r for all half prolate spheroids that just contain the cylinder, as illustrated in the two examples above. The result is a lower bound for the minimum possible Q_r for the cylindrical shape, i.e., min Q_r cylinder > max (min Q(0,2)) for all containing prolate spheroids. In Figure 11, the resulting curve for the lower bound of min Q_r cylinder versus aspect ratio is labeled max (min Q(0,2)). This curve is a lower bound for the minimum possible Q_r for a cylinder as a function of aspect ratio.

In paper by Hansen and Adams (2005) at the IEEE APS, the lowest order mode, Q(0), for the minimum volume half prolate spheroid was used to estimate the minimum Q_r for a cylinder. At that time, the minimum Q(0,2) was unavailable. Applying the minimum volume concept but using the new results, including higher order modes and minimum Q(0,2), provides the results shown as the middle curve in Figure 11. In this case, the volume of the half prolate spheroid is always 1.732 times the volume of the cylinder. This value is another lower bound, but obviously not as tight as the maximum (min Q(0,2)).

The third curve in Figure 11 is the Chu minimum Q_r for a hemisphere that just contains the cylinder. The prolate spheroid calculations include this case when b/a => 1. This curve is also lower than the lower bound determined from max(min Qmin (0,2)), except as h/r approaches 1, where they converge.

At high aspect ratios, the radius of the hemisphere that just contains the cylinder approaches the height of the cylinder so the ratio of Q_s goes to 1. When the cylinder aspect ratio approaches 1, the shape of the half prolate spheroid with maximum (min Q(0,2)) approaches the hemisphere that just contains the cylinder so that the normalized Qs of both curves converge. Near cylinder aspect ratios of 1, the new bound based on the prolate spheroid gives essentially the same results as Chu's formula. However, at higher cylinder aspect ratios, the new bound is much higher than the bound given by Chu.

8.2.2 Comparison with Practical Antennas

The radiation Q_r has been calculated for several monopole-type antennas that fit in a cylinder of fixed height (h) of 0.01 λ above a PC ground as a function of aspect ratio (h/r). The antennas considered are a thin cylinder (TC), fat monopole (FM), conical monopole (CM), umbrella top-loaded monopole (UTLM), disk-loaded monopole (DLM), and the minimum Q Wheeler inverted cup (WIC) (Hansen and Adams, 2005).

MININEC was used to generate the data for a 36-wire model of the CM and WIC, as well as the data for a UTLM with 18 top-load radials at 45 degrees. NEC-4 generated the data for a 36-wire FM. The DLM data are from Simpson (2004) and the TC data are from Tai (1984).

Figure 12 shows curves of the radiation Q_r of these antennas as a function of aspect ratio, normalized to the Chu minimum Q_r for a sphere with a 0.01 λ radius, and the lower bound for Qr for a cylinder (max (min Q(0,2)).

Several interesting points are illustrated in Figure 12. The radiation Q of the practical antennas examined is everywhere above the lower bound for a cylinder determined from the prolate spheroid formula. The curve of Q_r for the UTLM has a minimum when h/r = 1.4. This type of antenna has a minimum Q_r when h'/h ≈ 0.7, with cone angles near 45° (Devaney, et al., 1966).

At large aspect ratios, Q_r for the FM, CM, and DLM appear to be converging to that for the thin cylinder. As the aspect ratio decreases the value of Q_r for the FM, the CM, DLM, and WIC become increasingly small. Q_r falls below the Chu minimum for a hemisphere of the same height because the volume within the cylinder is greater than the volume of the hemisphere.

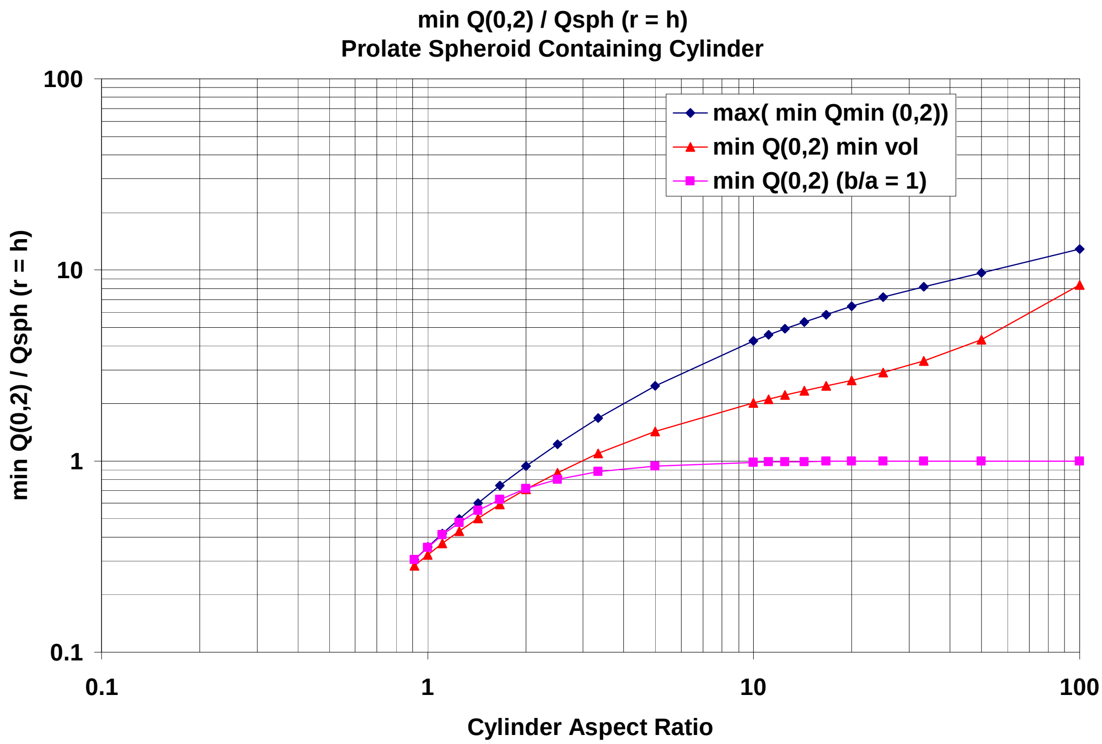


Figure 11. Lower bound for minimum Q_r of cylinder versus aspect ratio.

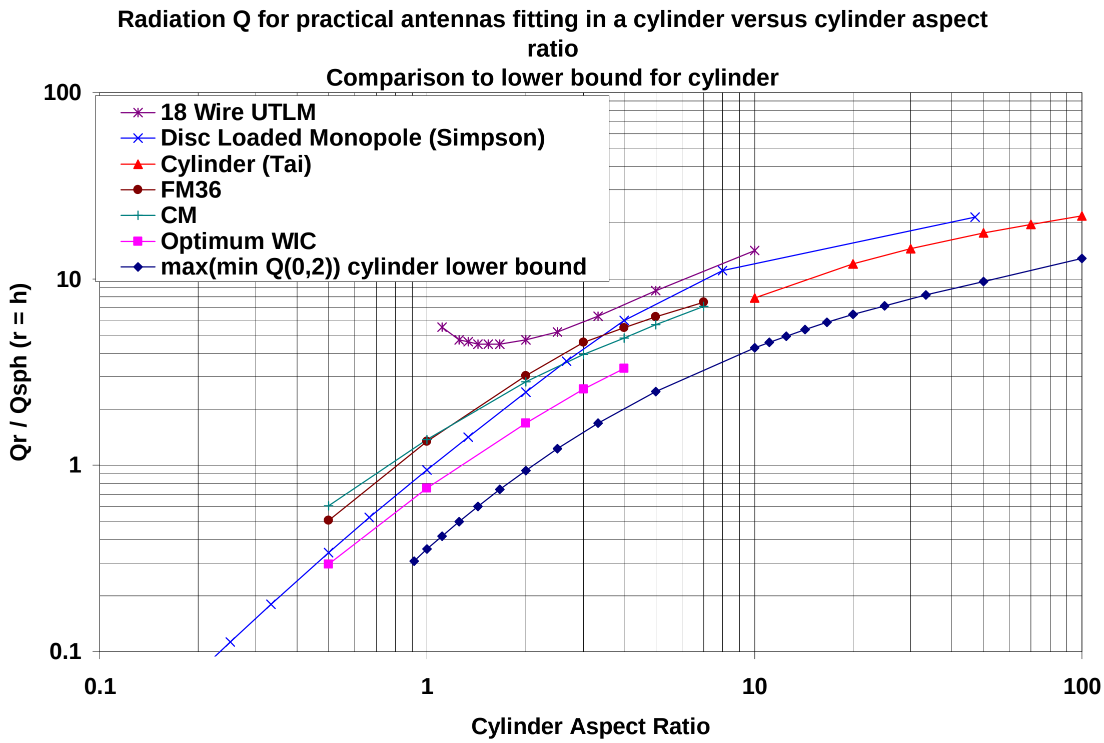


Figure 12. Radiation Q for practical antennas fitting in a cylinder compared to the new lower bound for cylinder (antennas 0.01λ tall, kh = 0.0628).

The bound derived from the prolate spheroid formula does not apply when the cylinder aspect ratio is much below 1, and a similar bound for the oblate spheroid case needs development. The WIC has the lowest Q_r of any antenna examined. For an aspect ratio of 1, it is equal to 0.76 times the Chu limit for a hemisphere of the same height.

9. CONCLUSIONS AND RECOMMENDATIONS

The minimum possible radiation Q was calculated numerically for an electrically small antenna enclosed in a prolate spheroid as a function of aspect ratio. A simple but accurate formula that gives the minimum possible Q for aspect ratios from 1 to 100 was also developed. The prolate spheroid formula was used to determine a lower bound for the radiation Q for an electrically small antenna contained within a cylinder. The cylinder bound and the Chu bound are essentially the same for aspect ratios near 1. For large aspect ratios, the new bound indicates considerably higher minimum radiation Qs than the minimum radiation Qs resulting from Chu's formula.

The new bound has been compared to the radiation Q calculated for several practical antennas. The results are consistent in that the new bound is always less than the radiation Q calculated for the practical antennas. Lower radiation Qs are possible for cylindrical-shaped antennas with aspect ratios less than 1 corresponding to the oblate spheroid case.

The oblate spheroidal case covers antennas that have aspect ratios less than 1. Oblate shapes are wider than they are high. This type of antenna has a smaller radiation Q than high aspect ratio antennas. For high-power, electrically small transmitting antennas, the oblate shape has great potential for practical use. Therefore, it is recommended that the analysis for the oblate spheroid case be completed and the results compared to practical antennas.

10. REFERENCES

Abramowitz, M. and I. A. Stegun. 1965. *Handbook of Mathematical Functions*, Dover Publications, Mineola, NY.

Adams, R. C. and P. M. Hansen. 2004. “Evaluation of Q in an Electrically Small Antenna in Prolate Spheroidal Coordinates.” Technical Document 3188, SPAWAR Systems Center San Diego, San Diego, CA.

Adams, R. C. and P. M. Hansen. 2005. “Evaluation of the Quality Factor of an Electrically Small Antenna in Spheroidal Coordinates,” IEEE International Symposium on Antennas and Propagation, 3–8 July, Washington, DC.

Chu, L. J. 1948.“Physical Limitations of Omni-Directional Antennas,” *Journal of Applied Physics*, volume 19 (December), p. 1163.

Devaney, T. E., R. F. Hall, and W. E. Gustafson. 1966. “Low-Frequency Top-loaded Antennas.” TR 1381 (June), Navy Electronics Laboratory*, San Diego, CA.

Dwight, H. B. 1961. *Tables of Integrals and Other Mathematical Data*. Macmillan Company, New York, NY.

Flammer, C. 1957. *Spheroidal Wave Functions*. Stanford University Press, Stanford, CA.

Hansen, P. M. and R. C. Adams. 2005. “Radiation Q of Practical Electrically Small Antennas as a Function of Aspect Ratio–Comparison to Minimum Q Formulas,” IEEE International Symposium on Antennas and Propagation, 3–8 July, Washington DC.

McLean, J. S. 1996. “A Re-examination of the Fundamental Limits on the Radiation Q of Electrically Small Antennas,” *IEEE Transactions on Antennas and Propagation*, vol. AP-44, no. 5 (May), pp. 672–676.

Simpson, T. L. 2004. “The Disk Loaded Monopole Antenna,” *IEEE Transactions on Antennas and Propagation*, vol. 52, No. 2 (Feb), pp. 542–550.

Tai, C-T. 1984. “Dipoles and Monopoles.” In *Antenna Engineering Handbook*, Chapter 4, p. 4-4 to 4-34, R. C. Johnson and H. Jasik, Eds. McGraw Hill Publishing, New York, NY.

* now SSC San Diego

APPENDIX A

NUMERICAL METHODS, RESULTS, AND COMPARISON WITH FLAMMER'S NUMBERS

Probably the most important quantity to be calculated is the eigenvalue of the angular eigen equation. The equation to be solved is provided by Equation (4a) in the body of this report:

$$0 = ((1-\eta^2)S^{(n)}{}_\eta)_\eta + (\kappa^{(n)} - \beta^2 \eta^2)S^{(n)} \quad \text{(A1)}$$

The requirement that $S^{(n)}(\beta,\eta)$ be finite at $\eta = \pm 1$ is satisfied only if $\kappa^{(n)}$ is equal to one of a particular set of values, the eigenvalues. A power series in β has been given in several books (Flammer [1957], p. 18; or Abramowitz and Stegun (1965, p. 754). The formula is

$$\begin{aligned}\kappa^{(n)} =& n(n+1) + 0.5\cdot\beta^2 [1-(2n-1)^{-1}(2n+3)^{-1}]+\beta^4[(n-1)^2n^2(2n-3)^{-1}(2n-1)^{-3}(2n+1)^{-1} - \\ & (n+1)^2(n+2)^2(2n+1)^{-1}(2n+3)^{-2}(2n+5)^{-1}] + \\ & \beta^6[(n+1)^2(n+2)^2(2n-1)^{-1}(2n+1)^{-1}(2n+3)^{-5} (2n+5)^{-1}(2n+7)^{-1} - \\ & (n-1)^2n^2(2n-5)^{-1}(2n-3)^{-1}(2n-1)^{-5}(2n+1)^{-1}(2n+3)^{-1}] +O(\beta^8)\end{aligned} \quad \text{(A2)}$$

We obtained five-place accuracy when compared with Flammer's results. Table A-1 shows the eigenvalue results. Cases in which comparison with Flammer's tables were made are presented in bold type.

Table A-1. Eigenvalues for angular function.

β\Mode	$\kappa^{(0)}$	$\kappa^{(1)}$	$\kappa^{(2)}$	$\kappa^{(3)}$	$\kappa^{(4)}$	$\kappa^{(5)}$	$\kappa^{(6)}$
0.0	**0.000000**	**2.000000**	**6.000000**	**12.000000**	20.000000	30.000000	42.000000
0.1	0.003330	**2.005999**	**6.005239**	**12.005111**	20.005065	30.005043	42.005030
0.2	**0.013310**	**2.023989**	**6.020969**	**12.020450**	20.020263	30.020173	42.020122
0.3	0.029831	2.053944	6.047225	12.046027	20.045599	30.045394	42.045279
0.4	**0.052960**	**2.095824**	**6.084067**	**12.081862**	20.081084	30.080713	42.080505
0.5	**0.082420**	**2.149569**	**6.131579**	**12.127985**	20.126734	30.126139	42.125807
0.6	**0.118100**	**2.215105**	**6.189864**	**12.184430**	20.182568	30.181685	42.181193
0.7	0.159556	2.292337	6.259047	12.251242	20.248609	30.247365	42.246674
0.8	**0.207390**	**2.381154**	**6.339269**	**12.328476**	20.324884	30.323198	42.322262
0.9	0.260095	2.481425	6.430688	12.416192	20.411427	30.409203	42.407971
1.0	**0.319000**	**2.593001**	**6.533476**	**12.514463**	20.508273	30.505404	42.503818

Equation (A1) was solved by using a power series in η^2. Since the eigenvalue is known and the initial values of the function and the derivative are known, the use of a power series method with recursion relations is straightforward. The power series method gave good results (when compared with Flammer) as long as the value of β was less than 1.8.

For values of β larger than 3, the results of the power series method differed significantly from Flammer. For these larger values of β, the angular eigenfunction needed to be expanded in a series of Legendre polynomials. Expansion was performed with good results, and the power series method was used for all the computations presented in this report.

Since a method to compute the eigenfunction now exists, the needed normalization constants can be computed. Simpson's rule was used (with an increment of η of 0.01 201

points for the interval). These results were then compared with those computed using 2001 data, with no difference for the five places printed.

Table A-2 presents the calculation of a normalization constant needed in evaluating the radial function. Values in bold print show those compared with Flammer's results. The formulas for the computation of the normalization constants are

$$d^{(n)}(\beta) = (2/(2n+1)) \int_{-1}^{1} d\eta\, S^{(n)}(\eta, \beta) \quad \text{for n even} \tag{A3a}$$

or

$$d^{(n)}(\beta) = (2/(2n+1)) \int_{-1}^{1} d\eta\, \eta\, S^{(n)}(\eta, \beta) \quad \text{for n odd.} \tag{A3b}$$

Table A-2. Calculation of the normalization constants for radial eigenfunction.

β\Mode	$d^{(0)}$	$d^{(1)}$	$d^{(2)}$	$d^{(3)}$	$d^{(4)}$	$d^{(5)}$
0.0	**1.000E+00**	**1.000E+00**	**0.000E+00**	**0.000E+00**	0.000E+00	0.000E+00
0.1	**9.9945E-01**	**9.9940E-01**	**2.2221E-04**	**1.7143E-04**	9.0702E-09	6.8714E-09
0.2	**9.9778E-01**	**9.9760E-01**	**8.8868E-04**	**6.8567E-04**	1.4512E-07	1.0994E-07
0.3	9.9505E-01	9.9462E-01	1.9989E-03	1.5426E-03	7.3464E-07	5.5656E-07
0.4	**9.9121E-01**	**9.9046E-01**	**3.5521E-03**	**2.7421E-03**	2.3217E-06	1.7590E-06
0.5	**9.8636E-01**	**9.8514E-01**	**5.5469E-03**	**4.2838E-03**	5.6677E-06	4.2941E-06
0.6	**9.8053E-01**	**9.7869E-01**	**7.9812E-03**	**6.1673E-03**	1.1751E-05	8.9038E-06
0.7	9.7382E-01	9.7113E-01	1.0852E-02	8.3922E-03	2.1768E-05	1.6494E-05
0.8	**9.6608E-01**	**9.6250E-01**	**1.4157E-02**	**1.0958E-02**	3.7130E-05	2.8136E-05
0.9	9.5773E-01	9.5284E-01	1.7890E-02	1.3863E-02	5.9465E-05	4.5065E-05
1.0	**9.4837E-01**	**9.4219E-01**	**2.2045E-02**	**1.7108E-02**	9.0614E-05	6.8678E-05

Flammer also presented the eigenfunction values and their derivatives for numerous values of the angular coordinate (η = cos(θ), where the increment in θ is 5°). Table A-3 presents the angular function.

Table A-3. Values of angular eigenfunction for six modes (four β values and 19 values of θ).

Θ	β = 0.25 $S^{(0)}(\beta,\eta)$	$S^{(1)}(\beta,\eta)$	$S^{(2)}(\beta,\eta)$	$S^{(3)}(\beta,\eta)$	$S^{(4)}(\beta,\eta)$	$S^{(5)}(\beta,\eta)$
0	9.89736E-01	9.93768E-01	1.00148E+00	9.99307E-01	1.00041E+00	9.99733E-01
5	9.89798E-01	9.90034E-01	9.90127E-01	9.76656E-01	9.62720E-01	9.43482E-01
10	9.90011E-01	9.78855E-01	9.56403E-01	9.10156E-01	8.53770E-01	7.83988E-01
15	9.90378E-01	9.60309E-01	9.01321E-01	8.04063E-01	6.85403E-01	5.47373E-01
20	9.90886E-01	9.34520E-01	8.26535E-01	6.65151E-01	4.75815E-01	2.71963E-01
25	9.91516E-01	9.01667E-01	7.34292E-01	5.02264E-01	2.47433E-01	1.47416E-03
30	9.92249E-01	8.61976E-01	6.27369E-01	3.25718E-01	2.43013E-02	-2.22696E-01
35	9.93062E-01	8.15724E-01	5.08990E-01	1.46608E-01	-1.70710E-01	-3.68679E-01
40	9.93929E-01	7.63241E-01	3.82732E-01	-2.39336E-02	-3.18534E-01	-4.19518E-01
45	9.94825E-01	7.04901E-01	2.52418E-01	-1.75488E-01	-4.06078E-01	-3.75765E-01
50	9.95722E-01	6.41130E-01	1.22003E-01	-2.99052E-01	-4.27658E-01	-2.54833E-01
55	9.96592E-01	5.72399E-01	-4.54656E-03	-3.87644E-01	-3.85550E-01	-8.72662E-02
60	9.97410E-01	4.99220E-01	-1.23371E-01	-4.36773E-01	-2.89567E-01	8.93559E-02
65	9.98149E-01	4.22147E-01	-2.30842E-01	-4.44736E-01	-1.55742E-01	2.37703E-01
70	9.98787E-01	3.41770E-01	-3.23667E-01	-4.12729E-01	-4.25734E-03	3.27796E-01
75	9.99305E-01	2.58711E-01	-3.99000E-01	-3.44756E-01	1.43112E-01	3.42589E-01

80	9.99687E-01	1.73615E-01	-4.54528E-01	-2.47341E-01	2.65739E-01	2.80970E-01
85	9.99921E-01	8.71516E-02	-4.88544E-01	-1.29073E-01	3.46723E-01	1.57657E-01
90	1.00000E+00	6.12323E-17	-5.00000E-01	-9.18485E-17	3.75000E-01	1.14811E-16

β = 0.5

$\Theta = \cos^{-1}\eta$	$S^{(0)}(\beta,\eta)$	$S^{(1)}(\beta,\eta)$	$S^{(2)}(\beta,\eta)$	$S^{(3)}(\beta,\eta)$	$S^{(4)}(\beta,\eta)$	$S^{(5)}(\beta,\eta)$
0	**9.59287E-01**	**9.75294E-01**	**1.00586E+00**	**9.97241E-01**	1.00162E+00	9.98934E-01
5	9.59595E-01	9.71768E-01	9.94622E-01	9.74808E-01	9.64061E-01	9.42899E-01
10	9.60505E-01	9.61204E-01	9.61236E-01	9.08929E-01	8.55450E-01	7.83982E-01
15	9.61990E-01	9.43646E-01	9.06664E-01	8.03764E-01	6.87524E-01	5.48114E-01
20	9.64005E-01	9.19170E-01	8.32482E-01	6.65952E-01	4.78328E-01	2.73374E-01
25	9.66492E-01	8.87886E-01	7.40846E-01	5.04173E-01	2.50142E-01	3.25811E-03
30	**9.69378E-01**	**8.49942E-01**	**6.34437E-01**	**3.28587E-01**	2.69004E-02	-2.20966E-01
35	9.72576E-01	8.05532E-01	5.16387E-01	1.50164E-01	-1.68557E-01	-3.67422E-01
40	9.75992E-01	7.54897E-01	3.90202E-01	-2.00398E-02	-3.17113E-01	-4.19022E-01
45	9.79522E-01	6.98333E-01	2.59665E-01	-1.71626E-01	-4.05550E-01	-3.76104E-01
50	9.83060E-01	6.36191E-01	1.28721E-01	-2.95549E-01	-4.28020E-01	-2.55864E-01
55	9.86498E-01	5.68886E-01	1.36701E-03	-3.84741E-01	-3.86631E-01	-8.86873E-02
60	**9.89729E-01**	**4.96891E-01**	**-1.18475E-01**	**-4.34591E-01**	-2.91075E-01	8.78943E-02
65	9.92656E-01	4.20740E-01	-2.27083E-01	-4.43277E-01	-1.57335E-01	2.36490E-01
70	9.95186E-01	3.41024E-01	-3.21058E-01	-4.11892E-01	-5.62850E-03	3.26984E-01
75	9.97242E-01	2.58387E-01	-3.97438E-01	-3.44370E-01	1.42161E-01	3.42171E-01
80	9.98758E-01	1.73518E-01	-4.53802E-01	-2.47220E-01	2.65252E-01	2.80829E-01
85	9.99687E-01	8.71392E-02	-4.88358E-01	-1.29057E-01	3.46591E-01	1.57638E-01
90	**1.00000E+00**	**6.12323E-17**	**-5.00000E-01**	**-9.18485E-17**	3.75000E-01	1.14811E-16

β = 0.75

$\Theta = \cos^{-1}\eta$	$S^{(0)}(\beta,\eta)$	$S^{(1)}(\beta,\eta)$	$S^{(2)}(\beta,\eta)$	$S^{(3)}(\beta,\eta)$	$S^{(4)}(\beta,\eta)$	$S^{(5)}(\beta,\eta)$
0	9.11721E-01	9.45235E-01	1.01290E+00	9.93842E-01	1.00364E+00	9.97608E-01
5	9.12253E-01	9.42041E-01	1.00186E+00	9.71769E-01	9.66289E-01	9.41932E-01
10	9.14060E-01	9.32464E-01	9.69053E-01	9.06914E-01	8.58246E-01	7.83973E-01
15	9.17163E-01	9.16497E-01	9.15346E-01	8.03285E-01	6.91059E-01	5.49348E-01
20	9.21445E-01	8.94137E-01	8.42194E-01	6.67290E-01	4.82524E-01	2.75726E-01
25	9.26772E-01	8.65386E-01	7.51600E-01	5.07345E-01	2.54671E-01	6.23197E-03
30	9.32982E-01	8.30270E-01	6.46079E-01	3.33352E-01	3.12574E-02	-2.18077E-01
35	9.39894E-01	7.88845E-01	5.28615E-01	1.56073E-01	-1.64935E-01	-3.65317E-01
40	9.47302E-01	7.41216E-01	4.02591E-01	-1.35678E-02	-3.14705E-01	-4.18179E-01
45	9.54983E-01	6.87546E-01	2.71716E-01	-1.65201E-01	-4.04630E-01	-3.76652E-01
50	9.62706E-01	6.28068E-01	1.39919E-01	-2.89718E-01	-4.28587E-01	-2.57565E-01
55	9.70232E-01	5.63100E-01	1.12448E-02	-3.79902E-01	-3.88407E-01	-9.10446E-02
60	9.77328E-01	4.93049E-01	-1.10281E-01	-4.30953E-01	-2.93572E-01	8.54647E-02
65	9.83769E-01	4.18415E-01	-2.20783E-01	-4.40843E-01	-1.59982E-01	2.34471E-01
70	9.89351E-01	3.39790E-01	-3.16679E-01	-4.10493E-01	-7.91125E-03	3.25630E-01
75	9.93893E-01	2.57851E-01	-3.94813E-01	-3.43726E-01	1.40575E-01	3.41475E-01
80	9.97248E-01	1.73356E-01	-4.52582E-01	-2.47017E-01	2.64440E-01	2.80593E-01
85	9.99306E-01	8.71188E-02	-4.88044E-01	-1.29031E-01	3.46370E-01	1.57606E-01
90	1.00000E+00	6.12323E-17	-5.00000E-01	-9.18485E-17	3.75000E-01	1.14811E-16

β = 1.00

$\Theta = \cos^{-1}\eta$	$S^{(0)}(\beta,\eta)$	$S^{(1)}(\beta,\eta)$	$S^{(2)}(\beta,\eta)$	$S^{(3)}(\beta,\eta)$	$S^{(4)}(\beta,\eta)$	$S^{(5)}(\beta,\eta)$
0	**8.48142E-01**	**9.04676E-01**	**1.02219E+00**	**9.89170E-01**	1.00644E+00	9.95765E-01
5	**8.49239E-01**	**9.01915E-01**	**1.01144E+00**	**9.67594E-01**	9.69393E-01	9.40589E-01
10	**8.52502E-01**	**8.93635E-01**	**9.79453E-01**	**9.04154E-01**	8.62150E-01	7.83965E-01
15	**8.57847E-01**	**8.79777E-01**	**9.26991E-01**	**8.02646E-01**	6.96007E-01	5.51073E-01
20	**8.65133E-01**	**8.60228E-01**	**8.55331E-01**	**6.69165E-01**	4.88413E-01	2.79014E-01
25	**8.74165E-01**	**8.34856E-01**	**7.66258E-01**	**5.11764E-01**	2.61044E-01	1.03967E-02
30	**8.84699E-01**	**8.03524E-01**	**6.62057E-01**	**3.39986E-01**	3.74096E-02	-2.14020E-01
35	**8.96441E-01**	**7.66110E-01**	**5.45493E-01**	**1.64302E-01**	-1.59794E-01	-3.62345E-01
40	**9.09059E-01**	**7.22534E-01**	**4.19777E-01**	**-4.54452E-03**	-3.11254E-01	-4.16965E-01
45	**9.22183E-01**	**6.72780E-01**	**2.88505E-01**	**-1.56233E-01**	-4.03262E-01	-3.77382E-01
50	**9.35421E-01**	**6.16922E-01**	**1.55580E-01**	**-2.81567E-01**	-4.29310E-01	-2.59915E-01
55	**9.48366E-01**	**5.55143E-01**	**2.51044E-02**	**-3.73131E-01**	-3.90837E-01	-9.43213E-02
60	**9.60611E-01**	**4.87755E-01**	**-9.87513E-02**	**-4.25855E-01**	-2.97032E-01	8.20770E-02
65	**9.71761E-01**	**4.15205E-01**	**-2.11898E-01**	**-4.37429E-01**	-1.63670E-01	2.31651E-01
70	**9.81449E-01**	**3.38082E-01**	**-3.10491E-01**	**-4.08528E-01**	-1.11018E-02	3.23735E-01
75	**9.89351E-01**	**2.57110E-01**	**-3.91097E-01**	**-3.42820E-01**	1.38353E-01	3.40501E-01
80	**9.95198E-01**	**1.73131E-01**	**-4.50852E-01**	**-2.46731E-01**	2.63300E-01	2.80262E-01
85	**9.98789E-01**	**8.70903E-02**	**-4.87599E-01**	**-1.28994E-01**	3.46061E-01	1.57561E-01
90	**1.00000E+00**	**6.12323E-17**	**-5.00000E-01**	**-9.18485E-17**	3.75000E-01	1.14811E-16

A second set of normalization constants is needed for computing expansion coefficients. The angular eigenfunctions computed by the power series method and Simpson's rule were used again to compute the integrals. Table A-4 presents these computations. For $\beta = 0.5$ and mode 1, the result differed with Flammer's in the second place. The inaccuracy of Flammer's result was pointed out in a phone conversation from a former SSC San Diego employee.

$$J^{(n)}(\beta) = \int_{-1}^{1} d\,\eta\; S^{(n)}(\eta, \beta)^2 \qquad \text{(A4)}$$

Table A-4. Normalization constants for angular eigenfunctions.

β\Mode	$J^{(0)}$	$J^{(1)}$	$J^{(2)}$	$J^{(3)}$	$J^{(4)}$	$J^{(5)}$
0.0	2.0000E+00	6.6667E-01	4.0000E-01	2.8571E-01	2.2222E-01	1.8182E-01
0.1	1.9978E+00	6.6587E-01	4.0021E-01	2.8565E-01	2.2225E-01	1.8180E-01
0.2	1.9912E+00	6.6348E-01	4.0084E-01	2.8547E-01	2.2234E-01	1.8176E-01
0.3	1.9803E+00	6.5951E-01	4.0188E-01	2.8517E-01	2.2249E-01	1.8168E-01
0.4	1.9651E+00	6.5401E-01	4.0336E-01	2.8475E-01	2.2270E-01	1.8158E-01
0.5	**1.9461E+00**	*6.4703E-01*	**4.0526E-01**	**2.8422E-01**	2.2296E-01	1.8144E-01
0.6	1.9235E+00	6.3861E-01	4.0759E-01	2.8357E-01	2.2329E-01	1.8127E-01
0.7	1.8978E+00	6.2883E-01	4.1037E-01	2.8281E-01	2.2368E-01	1.8108E-01
0.8	1.8684E+00	6.1778E-01	4.1360E-01	2.8195E-01	2.2412E-01	1.8085E-01
0.9	1.8373E+00	6.0555E-01	4.1729E-01	2.8100E-01	2.2463E-01	1.8060E-01
1.0	**1.8030E+00**	**5.9223E-01**	**4.2144E-01**	**2.7994E-01**	2.2519E-01	1.8032E-01

Table A-5. Derivatives of angular eigenfunction with respect to η = cos(θ) for six modes (four values of β and 19 values of θ).

$\beta = 0.25$

θ	$S_\eta^{(0)}(\beta,\theta)$	$S_\eta^{(1)}(\beta,\theta)$	$S_\eta^{(2)}(\beta,\theta)$	$S_\eta^{(3)}(\beta,\theta)$	$S_\eta^{(4)}(\beta,\theta)$	$S_\eta^{(5)}(\beta,\theta)$
0	-2.0334E-02	9.8133E-01	2.9896E+00	5.9806E+00	9.9886E+00	1.4981E+01
5	-2.0268E-02	9.8147E-01	2.9784E+00	5.9241E+00	9.8188E+00	1.4585E+01
10	-2.0068E-02	9.8189E-01	2.9449E+00	5.7563E+00	9.3193E+00	1.3437E+01
15	-1.9726E-02	9.8258E-01	2.8894E+00	5.4822E+00	8.5204E+00	1.1649E+01
20	-1.9237E-02	9.8351E-01	2.8122E+00	5.1100E+00	7.4698E+00	9.3913E+00
25	-1.8597E-02	9.8466E-01	2.7138E+00	4.6509E+00	6.2298E+00	6.8777E+00
30	-1.7806E-02	9.8599E-01	2.5948E+00	4.1186E+00	4.8735E+00	4.3381E+00
35	-1.6871E-02	9.8746E-01	2.4561E+00	3.5291E+00	3.4794E+00	1.9941E+00
40	-1.5799E-02	9.8903E-01	2.2986E+00	2.9003E+00	2.1265E+00	3.5116E-02
45	-1.4601E-02	9.9065E-01	2.1234E+00	2.2511E+00	8.8885E-01	-1.4019E+00
50	-1.3286E-02	9.9227E-01	1.9318E+00	1.6013E+00	-1.6905E-01	-2.2462E+00
55	-1.1865E-02	9.9384E-01	1.7251E+00	9.7061E-01	-9.9683E-01	-2.4983E+00
60	-1.0351E-02	9.9532E-01	1.5048E+00	3.7828E-01	-1.5614E+00	-2.2271E+00
65	-8.7543E-03	9.9666E-01	1.2728E+00	-1.5756E-01	-1.8491E+00	-1.5588E+00
70	-7.0884E-03	9.9781E-01	1.0306E+00	-6.2046E-01	-1.8664E+00	-6.5858E-01
75	-5.3662E-03	9.9875E-01	7.8023E-01	-9.9618E-01	-1.6396E+00	2.9189E-01
80	-3.6014E-03	9.9944E-01	5.2365E-01	-1.2732E+00	-1.2124E+00	1.1185E+00
85	-1.8079E-03	9.9986E-01	2.6288E-01	-1.4428E+00	-6.4307E-01	1.6777E+00
90	-1.2702E-18	1.0000E+00	1.8470E-16	-1.5000E+00	-4.5997E-16	1.8750E+00

$\beta = 0.50$

θ	$S_\eta^{(0)}(\beta,\theta)$	$S_\eta^{(1)}(\beta,\theta)$	$S_\eta^{(2)}(\beta,\theta)$	$S_\eta^{(3)}(\beta,\theta)$	$S_\eta^{(4)}(\beta,\theta)$	$S_\eta^{(5)}(\beta,\theta)$
0	-8.0434E-02	9.2632E-01	2.9580E+00	5.9226E+00	9.9545E+00	1.4922E+01
5	-8.0142E-02	9.2687E-01	2.9476E+00	5.8675E+00	9.7864E+00	1.4530E+01
10	-7.9266E-02	9.2851E-01	2.9162E+00	5.7038E+00	9.2921E+00	1.3391E+01
15	-7.7811E-02	9.3119E-01	2.8641E+00	5.4361E+00	8.5008E+00	1.1616E+01
20	-7.5786E-02	9.3483E-01	2.7914E+00	5.0721E+00	7.4592E+00	9.3728E+00
25	-7.3200E-02	9.3932E-01	2.6982E+00	4.6222E+00	6.2283E+00	6.8731E+00
30	-7.0067E-02	9.4453E-01	2.5849E+00	4.0993E+00	4.8799E+00	4.3444E+00
35	-6.6402E-02	9.5032E-01	2.4519E+00	3.5187E+00	3.4915E+00	2.0070E+00
40	-6.2226E-02	9.5650E-01	2.2999E+00	2.8976E+00	2.1414E+00	5.0022E-02
45	-5.7562E-02	9.6288E-01	2.1295E+00	2.2545E+00	9.0377E-01	-1.3890E+00
50	-5.2438E-02	9.6929E-01	1.9419E+00	1.6087E+00	-1.5664E-01	-2.2378E+00
55	-4.6890E-02	9.7552E-01	1.7380E+00	9.8014E-01	-9.8859E-01	-2.4953E+00
60	-4.0955E-02	9.8137E-01	1.5193E+00	3.8810E-01	-1.5580E+00	-2.2287E+00
65	-3.4678E-02	9.8668E-01	1.2875E+00	-1.4888E-01	-1.8501E+00	-1.5633E+00
70	-2.8108E-02	9.9127E-01	1.0442E+00	-6.1383E-01	-1.8706E+00	-6.6373E-01
75	-2.1296E-02	9.9500E-01	7.9160E-01	-9.9195E-01	-1.6452E+00	2.8775E-01
80	-1.4301E-02	9.9775E-01	5.3180E-01	-1.2711E+00	-1.2175E+00	1.1162E+00
85	-7.1820E-03	9.9943E-01	2.6713E-01	-1.4423E+00	-6.4603E-01	1.6770E+00
90	-5.0468E-18	1.0000E+00	1.8773E-16	-1.5000E+00	-4.6215E-16	1.8750E+00

$\beta = 0.75$

θ	$S_\eta^{(0)}(\beta,\theta)$	$S_\eta^{(1)}(\beta,\theta)$	$S_\eta^{(2)}(\beta,\theta)$	$S_\eta^{(3)}(\beta,\theta)$	$S_\eta^{(4)}(\beta,\theta)$	$S_\eta^{(5)}(\beta,\theta)$
0	-1.7070E-01	8.3791E-01	2.9046E+00	5.8269E+00	9.8973E+00	1.4825E+01
5	-1.7020E-01	8.3910E-01	2.8954E+00	5.7742E+00	9.7322E+00	1.4438E+01
10	-1.6867E-01	8.4263E-01	2.8676E+00	5.6172E+00	9.2464E+00	1.3314E+01
15	-1.6606E-01	8.4841E-01	2.8211E+00	5.3601E+00	8.4678E+00	1.1560E+01
20	-1.6231E-01	8.5629E-01	2.7558E+00	5.0094E+00	7.4413E+00	9.3419E+00
25	-1.5735E-01	8.6604E-01	2.6713E+00	4.5746E+00	6.2256E+00	6.8653E+00
30	-1.5116E-01	8.7738E-01	2.5674E+00	4.0672E+00	4.8904E+00	4.3548E+00
35	-1.4377E-01	8.9000E-01	2.4441E+00	3.5014E+00	3.5115E+00	2.0285E+00
40	-1.3518E-01	9.0353E-01	2.3013E+00	2.8931E+00	2.1663E+00	7.4812E-02
45	-1.2546E-01	9.1756E-01	2.1392E+00	2.2600E+00	9.2867E-01	-1.3675E+00
50	-1.1464E-01	9.3168E-01	1.9583E+00	1.6210E+00	-1.3584E-01	-2.2239E+00
55	-1.0281E-01	9.4545E-01	1.7593E+00	9.9596E-01	-9.7474E-01	-2.4902E+00
60	-9.0037E-02	9.5844E-01	1.5434E+00	4.0443E-01	-1.5522E+00	-2.2314E+00
65	-7.6417E-02	9.7023E-01	1.3120E+00	-1.3441E-01	-1.8517E+00	-1.5706E+00
70	-6.2063E-02	9.8046E-01	1.0670E+00	-6.0277E-01	-1.8775E+00	-6.7229E-01
75	-4.7100E-02	9.8879E-01	8.1066E-01	-9.8488E-01	-1.6544E+00	2.8086E-01
80	-3.1667E-02	9.9495E-01	5.4549E-01	-1.2677E+00	-1.2259E+00	1.1124E+00
85	-1.5914E-02	9.9873E-01	2.7428E-01	-1.4414E+00	-6.5096E-01	1.6759E+00
90	-1.1186E-17	1.0000E+00	1.9281E-16	-1.5000E+00	-4.6580E-16	1.8750E+00

$\beta = 1.00$

Θ	$S_\eta^{(0)}(\beta,\theta)$	$S_\eta^{(1)}(\beta,\theta)$	$S_\eta^{(2)}(\beta,\theta)$	$S_\eta^{(3)}(\beta,\theta)$	$S_\eta^{(4)}(\beta,\theta)$	$S_\eta^{(5)}(\beta,\theta)$
0	-2.8879E-01	7.2086E-01	2.8281E+00	5.6949E+00	9.8170E+00	1.4690E+01
5	-2.8791E-01	7.2284E-01	2.8205E+00	5.6453E+00	9.6560E+00	1.4310E+01
10	-2.8527E-01	7.2875E-01	2.7977E+00	5.4975E+00	9.1820E+00	1.3207E+01
15	-2.8084E-01	7.3844E-01	2.7590E+00	5.2548E+00	8.4212E+00	1.1483E+01
20	-2.7459E-01	7.5168E-01	2.7040E+00	4.9226E+00	7.4156E+00	9.2986E+00
25	-2.6648E-01	7.6813E-01	2.6317E+00	4.5086E+00	6.2212E+00	6.8542E+00
30	-2.5647E-01	7.8736E-01	2.5412E+00	4.0226E+00	4.9046E+00	4.3691E+00
35	-2.4451E-01	8.0885E-01	2.4316E+00	3.4771E+00	3.5391E+00	2.0583E+00
40	-2.3059E-01	8.3198E-01	2.3019E+00	2.8865E+00	2.2009E+00	1.0940E-01
45	-2.1470E-01	8.5609E-01	2.1516E+00	2.2674E+00	9.6359E-01	-1.3374E+00
50	-1.9686E-01	8.8046E-01	1.9805E+00	1.6381E+00	-1.0653E-01	-2.2042E+00
55	-1.7713E-01	9.0434E-01	1.7886E+00	1.0180E+00	-9.5507E-01	-2.4830E+00
60	-1.5562E-01	9.2697E-01	1.5768E+00	4.2724E-01	-1.5438E+00	-2.2349E+00
65	-1.3247E-01	9.4760E-01	1.3463E+00	-1.1416E-01	-1.8538E+00	-1.5809E+00
70	-1.0786E-01	9.6556E-01	1.0991E+00	-5.8725E-01	-1.8871E+00	-6.8424E-01
75	-8.2022E-02	9.8022E-01	8.3759E-01	-9.7494E-01	-1.6673E+00	2.7122E-01
80	-5.5230E-02	9.9108E-01	5.6487E-01	-1.2629E+00	-1.2376E+00	1.1070E+00
85	-2.7782E-02	9.9775E-01	2.8441E-01	-1.4401E+00	-6.5789E-01	1.6744E+00
90	-1.9533E-17	1.0000E+00	2.0003E-16	-1.5000E+00	-4.7091E-16	1.8750E+00

We used the expression for the radial eigenfunction that involves the integral of the angular eigenfunction weighted by a spherical Hankel function. This expression is provided in the text as Equations (9a) or (9b). These equations are

$$R^{(n)}(\beta,u) = -j^{(n+1)} /[2\,\beta\, d^{(n)}(\beta)] \int_{-1}^{1} d\eta\, (u^2+\eta^2-1)^{-1/2} \exp[j\,\beta\,(u^2+\eta^2-1)^{1/2}]\, S^{(n)}(\beta,\eta) \qquad \text{(A5a)}$$

for n even, or

$$R^{(n)}(\beta,u) = -j^{n-1}\, 3u /[2\, d^{(n)}(\beta)] \int_{-1}^{1} d\eta\, \eta\, (u^2+\eta^2-1)^{-1} \{1+j[\beta(u^2+\eta^2-1)^{-1/2}]\} \exp[j\,\beta\,(u^2+\eta^2-1)^{1/2}]\, S^{(n)}(\beta,\eta) \qquad \text{(A5b)}$$

for n odd. The parameter $d^{(n)}$ is a normalization constant designed to ensure that the radial eigenfuctions have the correct value for large u. Table A-2 lists the values of d as a function of β. Simpson's rule was used to integrate for a series of values of u.

Table A-6 Values of real and imaginary parts of $R^{(n)}(\beta, u)$ for six modes (four values of β and numerous values of u).

β = 0.25

u\mode	$Re(R^{(0)})$	$Re(R^{(1)})$	$Re(R^{(2)})$	$Re(R^{(3)})$	$Re(R^{(4)})$	$Re(R^{(5)})$
1.005	9.9644E-01	8.3431E-02	2.8184E-03	6.0561E-05	1.5597E-02	3.7523E-02
1.010	9.9634E-01	8.3841E-02	2.8601E-03	6.2372E-05	1.5596E-02	3.7707E-02
1.015	9.9623E-01	8.4251E-02	2.9021E-03	6.4205E-05	1.5594E-02	3.7891E-02
1.020	9.9612E-01	8.4661E-02	2.9442E-03	6.6060E-05	1.5592E-02	3.8076E-02
1.025	9.9602E-01	8.5070E-02	2.9866E-03	6.7938E-05	1.5591E-02	3.8260E-02
1.030	9.9591E-01	8.5480E-02	3.0292E-03	6.9839E-05	1.5589E-02	3.8444E-02
1.035	9.9580E-01	8.5889E-02	3.0719E-03	7.1763E-05	1.5587E-02	3.8628E-02
1.040	9.9570E-01	8.6298E-02	3.1149E-03	7.3709E-05	1.5586E-02	3.8812E-02
1.044	9.9561E-01	8.6626E-02	3.1494E-03	7.5283E-05	1.5584E-02	3.8959E-02
1.077	9.9559E-01	8.9325E-02	3.4392E-03	8.8835E-05	1.5573E-02	4.0173E-02
2.000	9.9548E-01	1.6295E-01	1.5044E-02	9.9836E-04	1.5158E-02	7.3283E-02
2.250	9.9537E-01	1.8209E-01	1.9310E-02	1.4692E-03	1.5022E-02	8.1894E-02
2.500	9.9526E-01	2.0082E-01	2.4029E-02	2.0587E-03	1.4881E-02	9.0314E-02
2.750	9.9515E-01	2.1907E-01	2.9186E-02	2.7783E-03	1.4740E-02	9.8527E-02
3.000	9.9504E-01	2.3682E-01	3.4762E-02	3.6387E-03	1.4604E-02	1.0651E-01
3.250	9.9493E-01	2.5402E-01	4.0741E-02	4.6501E-03	1.4475E-02	1.1425E-01
3.500	9.9488E-01	2.7063E-01	4.7101E-02	5.8219E-03	1.4360E-02	1.2173E-01
3.750	9.9482E-01	2.8662E-01	5.3821E-02	7.1630E-03	1.4263E-02	1.2893E-01
4.000	9.9470E-01	3.0195E-01	6.0879E-02	8.6814E-03	1.4190E-02	1.3585E-01
4.250	9.9459E-01	3.1658E-01	6.8251E-02	1.0384E-02	1.4146E-02	1.4245E-01
4.500	9.9448E-01	3.3049E-01	7.5912E-02	1.2278E-02	1.4136E-02	1.4874E-01
4.750	9.9436E-01	3.4365E-01	8.3837E-02	1.4368E-02	1.4167E-02	1.5469E-01
5.000	9.9425E-01	3.5602E-01	9.1998E-02	1.6660E-02	1.4244E-02	1.6031E-01
6.000	9.9413E-01	3.9724E-01	1.2645E-01	2.7898E-02	1.5139E-02	1.7918E-01
87.000	9.9402E-01	4.2432E-01	1.6239E-01	4.2484E-02	1.7260E-02	1.9201E-01
0.000	9.9390E-01	4.3664E-01	1.9787E-01	6.0256E-02	2.0972E-02	1.9869E-01
9.000	9.9379E-01	4.3417E-01	2.3095E-01	8.0803E-02	2.6586E-02	1.9939E-01
10.000	9.9367E-01	4.1752E-01	2.5982E-01	1.0348E-01	3.4326E-02	1.9458E-01

$\beta = 0.5$

u\mode	$Re(R^{(0)})$	$Re(R^{(1)})$	$Re(R^{(2)})$	$Re(R^{(3)})$	$Re(R^{(4)})$	$Re(R^{(5)})$
1.005	**9.8595E-01**	**1.6497E-01**	**1.1258E-02**	**4.8989E-04**	8.2518E-04	4.4346E-03
1.010	9.8554E-01	1.6575E-01	1.1422E-02	5.0429E-04	8.2559E-04	4.4556E-03
1.015	9.8512E-01	1.6653E-01	1.1587E-02	5.1885E-04	8.2601E-04	4.4765E-03
1.020	**9.8470E-01**	**1.6730E-01**	**1.1753E-02**	**5.3360E-04**	8.2645E-04	4.4974E-03
1.025	9.8428E-01	1.6808E-01	1.1919E-02	5.4852E-04	8.2691E-04	4.5183E-03
1.030	9.8385E-01	1.6886E-01	1.2086E-02	5.6361E-04	8.2738E-04	4.5392E-03
1.035	9.8343E-01	1.6963E-01	1.2254E-02	5.7889E-04	8.2786E-04	4.5601E-03
1.040	9.8300E-01	1.7041E-01	1.2423E-02	5.9434E-04	8.2837E-04	4.5810E-03
1.044	**9.8265E-01**	**1.7103E-01**	**1.2559E-02**	**6.0683E-04**	8.2878E-04	4.5976E-03
1.077	**9.7977E-01**	**1.7612E-01**	**1.3695E-02**	**7.1431E-04**	8.3263E-04	4.7348E-03
2.000	8.6697E-01	3.0430E-01	5.7439E-02	7.7137E-03	1.5091E-03	8.2402E-03
2.250	8.2682E-01	3.3309E-01	7.2644E-02	1.1219E-02	2.0114E-03	9.0726E-03
2.500	7.8324E-01	3.5885E-01	8.8918E-02	1.5520E-02	2.7240E-03	9.8592E-03
2.750	7.3662E-01	3.8136E-01	1.0604E-01	2.0650E-02	3.6891E-03	1.0607E-02
3.000	6.8737E-01	4.0046E-01	1.2379E-01	2.6628E-02	4.9497E-03	1.1327E-02
3.250	6.3593E-01	4.1599E-01	1.4192E-01	3.3459E-02	6.5482E-03	1.2032E-02
3.500	5.8276E-01	4.2786E-01	1.6018E-01	4.1131E-02	8.5257E-03	1.2739E-02
3.750	5.2833E-01	4.3601E-01	1.7834E-01	4.9616E-02	1.0921E-02	1.3467E-02
4.000	4.7310E-01	4.4041E-01	1.9615E-01	5.8869E-02	1.3768E-02	1.4239E-02
4.250	4.1754E-01	4.4108E-01	2.1335E-01	6.8833E-02	1.7098E-02	1.5079E-02
4.500	3.6212E-01	4.3807E-01	2.2973E-01	7.9432E-02	2.0935E-02	1.6015E-02
4.750	3.0730E-01	4.3149E-01	2.4505E-01	9.0580E-02	2.5300E-02	1.7073E-02
5.000	2.5351E-01	4.2147E-01	2.5909E-01	1.0218E-01	3.0202E-02	1.8283E-02
6.000	5.6835E-02	3.5071E-01	2.9890E-01	1.5065E-01	5.5151E-02	2.5232E-02
7.000	-9.4398E-02	2.4333E-01	3.0629E-01	1.9594E-01	8.7482E-02	3.6816E-02
8.000	-1.8670E-01	1.1960E-01	2.7826E-01	2.2900E-01	1.2382E-01	5.4093E-02
9.000	-2.1721E-01	1.0054E-03	2.1858E-01	2.4211E-01	1.5892E-01	7.6884E-02
10.000	-1.9336E-01	-9.3720E-02	1.3702E-01	2.3073E-01	1.8655E-01	1.0344E-01

$\beta = 0.75$

u\mode	$Re(R^{(0)})$	$Re(R^{(1)})$	$Re(R^{(2)})$	$Re(R^{(3)})$	$Re(R^{(4)})$	$Re(R^{(5)})$
1.005	9.6910E-01	2.4281E-01	2.5259E-02	1.6526E-03	1.6918E-04	1.1925E-03
1.010	9.6817E-01	2.4387E-01	2.5619E-02	1.7005E-03	1.7297E-04	1.1980E-03
1.015	9.6724E-01	2.4494E-01	2.5979E-02	1.7490E-03	1.7684E-04	1.2034E-03
1.020	9.6631E-01	2.4601E-01	2.6342E-02	1.7980E-03	1.8080E-04	1.2088E-03
1.025	9.6537E-01	2.4707E-01	2.6705E-02	1.8476E-03	1.8485E-04	1.2143E-03
1.030	9.6443E-01	2.4813E-01	2.7071E-02	1.8978E-03	1.8898E-04	1.2197E-03
1.035	9.6348E-01	2.4919E-01	2.7437E-02	1.9486E-03	1.9320E-04	1.2251E-03
1.040	9.6253E-01	2.5024E-01	2.7805E-02	1.9999E-03	1.9751E-04	1.2306E-03
1.044	9.6177E-01	2.5109E-01	2.8101E-02	2.0414E-03	2.0102E-04	1.2349E-03
1.077	9.6157E-01	2.5799E-01	3.0574E-02	2.3979E-03	2.3225E-04	1.2707E-03
2.000	9.6062E-01	4.0593E-01	1.1943E-01	2.4497E-02	3.9365E-03	2.4773E-03
2.250	9.5965E-01	4.2828E-01	1.4725E-01	3.4947E-02	6.4176E-03	3.0243E-03
2.500	9.5869E-01	4.4223E-01	1.7513E-01	4.7305E-02	9.7973E-03	3.7744E-03
2.750	9.5771E-01	4.4768E-01	2.0224E-01	6.1436E-02	1.4199E-02	4.8030E-03

3.000	9.5674E-01	4.4471E-01	2.2775E-01	7.7129E-02	1.9722E-02	6.1934E-03
3.250	9.5576E-01	4.3361E-01	2.5087E-01	9.4098E-02	2.6438E-02	8.0335E-03
3.500	9.5537E-01	4.1486E-01	2.7091E-01	1.1199E-01	3.4382E-02	1.0413E-02
3.750	9.5478E-01	3.8910E-01	2.8721E-01	1.3041E-01	4.3548E-02	1.3418E-02
4.000	9.5379E-01	3.5714E-01	2.9927E-01	1.4889E-01	5.3887E-02	1.7128E-02
4.250	9.5280E-01	3.1991E-01	3.0666E-01	1.6697E-01	6.5304E-02	2.1614E-02
4.500	9.5181E-01	2.7843E-01	3.0911E-01	1.8414E-01	7.7656E-02	2.6926E-02
4.750	9.5081E-01	2.3383E-01	3.0646E-01	1.9992E-01	9.0758E-02	3.3098E-02
5.000	9.4980E-01	1.8725E-01	2.9873E-01	2.1382E-01	1.0438E-01	4.0141E-02
6.000	9.4880E-01	4.1403E-03	2.2147E-01	2.4257E-01	1.5831E-01	7.6222E-02
7.000	9.4779E-01	-1.2668E-01	9.4913E-02	2.1697E-01	1.9566E-01	1.1943E-01
8.000	9.4677E-01	-1.6846E-01	-3.4084E-02	1.3991E-01	1.9782E-01	1.5703E-01
9.000	9.4575E-01	-1.2445E-01	-1.1993E-01	3.5652E-02	1.5698E-01	1.7361E-01
10.000	9.4473E-01	-3.1626E-02	-1.3751E-01	-5.9878E-02	8.1533E-02	1.5779E-01

$\beta = 1.0$

u\mode	$Re(R^{(0)})$	$Re(R^{(1)})$	$Re(R^{(2)})$	$Re(R^{(3)})$	$Re(R^{(4)})$	$Re(R^{(5)})$
1.005	**9.4675E-01**	**3.1531E-01**	**4.4694E-02**	**3.9108E-03**	2.1824E-04	4.3891E-04
1.010	9.4513E-01	3.1655E-01	4.5307E-02	4.0221E-03	2.3045E-04	4.4151E-04
1.015	9.4350E-01	3.1779E-01	4.5922E-02	4.1347E-03	2.4293E-04	4.4414E-04
1.020	**9.4186E-01**	**3.1902E-01**	**4.6539E-02**	**4.2485E-03**	2.5567E-04	4.4679E-04
1.025	9.4022E-01	3.2025E-01	4.7159E-02	4.3637E-03	2.6868E-04	4.4948E-04
1.030	9.3857E-01	3.2148E-01	4.7780E-02	4.4800E-03	2.8196E-04	4.5218E-04
1.035	9.3691E-01	3.2270E-01	4.8404E-02	4.5977E-03	2.9550E-04	4.5492E-04
1.040	9.3525E-01	3.2392E-01	4.9030E-02	4.7166E-03	3.0933E-04	4.5768E-04
1.044	**9.3392E-01**	**3.2489E-01**	**4.9532E-02**	**4.8127E-03**	3.2058E-04	4.5991E-04
1.077	**9.2276E-01**	**3.3277E-01**	**5.3722E-02**	**5.6369E-03**	4.2038E-04	4.7908E-04
2.000	5.3226E-01	4.5604E-01	1.8955E-01	5.3252E-02	1.1400E-02	2.5874E-03
2.250	4.1453E-01	4.5428E-01	2.2509E-01	7.3875E-02	1.8326E-02	4.2684E-03
2.500	2.9900E-01	4.3787E-01	2.5640E-01	9.6875E-02	2.7395E-02	6.7898E-03
2.750	1.8928E-01	4.0801E-01	2.8179E-01	1.2139E-01	3.8683E-02	1.0362E-02
3.000	8.8611E-02	3.6643E-01	2.9985E-01	1.4640E-01	5.2135E-02	1.5181E-02
3.250	-2.7332E-04	3.1531E-01	3.0954E-01	1.7075E-01	6.7547E-02	2.1413E-02
3.500	-7.5223E-02	2.5713E-01	3.1022E-01	1.9327E-01	8.4572E-02	2.9171E-02
3.750	-1.3476E-01	1.9463E-01	3.0170E-01	2.1279E-01	1.0272E-01	3.8503E-02
4.000	-1.7809E-01	1.3059E-01	2.8422E-01	2.2820E-01	1.2136E-01	4.9374E-02
4.250	-2.0517E-01	6.7759E-02	2.5847E-01	2.3859E-01	1.3980E-01	6.1657E-02
4.500	-2.1658E-01	8.7049E-03	2.2551E-01	2.4319E-01	1.5723E-01	7.5128E-02
4.750	-2.1356E-01	-4.4291E-02	1.8675E-01	2.4154E-01	1.7284E-01	8.9463E-02
5.000	-1.9787E-01	-8.9333E-02	1.4386E-01	2.3341E-01	1.8585E-01	1.0425E-01
6.000	-5.6156E-02	-1.6894E-01	-3.1564E-02	1.4242E-01	1.9861E-01	1.5720E-01
7.000	8.9080E-02	-9.8195E-02	-1.3338E-01	3.0677E-03	1.3641E-01	1.7300E-01
8.000	1.2498E-01	3.0775E-02	-1.1349E-01	-1.0141E-01	2.4520E-02	1.2916E-01
9.000	4.9803E-02	1.0605E-01	-1.3232E-02	-1.1330E-01	-7.5022E-02	3.8257E-02
10.000	-5.1677E-02	8.0041E-02	7.6747E-02	-4.1829E-02	-1.0604E-01	-5.3681E-02

β = 0.25

u\mode	$Im(R^{(0)})$	$Im(R^{(1)})$	$Im(R^{(2)})$	$Im(R^{(3)})$	$Im(R^{(4)})$	$Im(R^{(5)})$
1.005	-1.1945E+01	-9.7525E+01	-2.2147E+03	-8.2687E+04	-4.2821E+06	-2.8352E+08
1.010	-1.0557E+01	-8.1439E+01	-1.7565E+03	-6.2526E+04	-3.0927E+06	-1.9579E+08
1.015	-9.7472E+00	-7.2220E+01	-1.5002E+03	-5.1552E+04	-2.4642E+06	-1.5085E+08
1.020	-9.1736E+00	-6.5800E+01	-1.3253E+03	-4.4234E+04	-2.0552E+06	-1.2234E+08
1.025	-8.7296E+00	-6.0904E+01	-1.1944E+03	-3.8868E+04	-1.7617E+06	-1.0234E+08
1.030	-8.3675E+00	-5.6968E+01	-1.0909E+03	-3.4704E+04	-1.5385E+06	-8.7437E+07
1.035	-8.0620E+00	-5.3691E+01	-1.0061E+03	-3.1352E+04	-1.3619E+06	-7.5864E+07
1.040	-7.7979E+00	-5.0895E+01	-9.3482E+02	-2.8578E+04	-1.2183E+06	-6.6614E+07
1.044	-7.6097E+00	-4.8924E+01	-8.8523E+02	-2.6675E+04	-1.1212E+06	-6.0453E+07
1.077	-6.5122E+00	-3.7861E+01	-6.1839E+02	-1.6887E+04	-6.4418E+05	-3.1543E+07
2.000	-1.9735E+00	-5.2332E+00	-3.1675E+01	-3.3460E+02	-4.9927E+03	-9.6037E+04
2.250	-1.6553E+00	-4.0825E+00	-2.1305E+01	-1.9584E+02	-2.5512E+03	-4.2885E+04
2.500	-1.4080E+00	-3.3091E+00	-1.5159E+01	-1.2332E+02	-1.4267E+03	-2.1324E+04
2.750	-1.2079E+00	-2.7594E+00	-1.1254E+01	-8.2023E+01	-8.5378E+02	-1.1494E+04
3.000	-1.0410E+00	-2.3517E+00	-8.6391E+00	-5.6955E+01	-5.3875E+02	-6.6003E+03
3.250	-8.9874E-01	-2.0390E+00	-6.8151E+00	-4.0952E+01	-3.5486E+02	-3.9888E+03
3.500	-7.7530E-01	-1.7923E+00	-5.5001E+00	-3.0311E+01	-2.4220E+02	-2.5149E+03
3.750	-6.6671E-01	-1.5931E+00	-4.5257E+00	-2.2992E+01	-1.7035E+02	-1.6432E+03
4.000	-5.7011E-01	-1.4288E+00	-3.7869E+00	-1.7813E+01	-1.2293E+02	-1.1071E+03
4.250	-4.8341E-01	-1.2910E+00	-3.2154E+00	-1.4057E+01	-9.0726E+01	-7.6593E+02
4.500	-4.0503E-01	-1.1733E+00	-2.7657E+00	-1.1274E+01	-6.8285E+01	-5.4240E+02
4.750	-3.3375E-01	-1.0715E+00	-2.4063E+00	-9.1749E+00	-5.2298E+01	-3.9212E+02
5.000	-2.6865E-01	-9.8209E-01	-2.1151E+00	-7.5644E+00	-4.0683E+01	-2.8875E+02
6.000	-5.7003E-02	-7.0785E-01	-1.3714E+00	-3.8952E+00	-1.6935E+01	-9.8734E+01
7.000	9.5762E-02	-5.1107E-01	-9.7879E-01	-2.2994E+00	-8.2642E+00	-4.0653E+01
8.000	2.0437E-01	-3.5525E-01	-7.4130E-01	-1.5048E+00	-4.5426E+00	-1.9187E+01
9.000	2.7710E-01	-2.2495E-01	-5.7948E-01	-1.0664E+00	-2.7436E+00	-1.0057E+01
10.000	3.1951E-01	-1.1349E-01	-4.5715E-01	-8.0273E-01	-1.7908E+00	-5.7268E+00

β = 0.5

u\mode	$Im(R^{(0)})$	$Im(R^{(1)})$	$Im(R^{(2)})$	$Im(R^{(3)})$	$Im(R^{(4)})$	$Im(R^{(5)})$
1.005	-5.9091E+00	-2.5079E+01	-2.7874E+02	-5.1857E+03	-1.3412E+05	-4.4372E+06
1.010	-5.2061E+00	-2.1008E+01	-2.2136E+02	-3.9243E+03	-9.6917E+04	-3.0656E+06
1.015	-4.7951E+00	-1.8675E+01	-1.8924E+02	-3.2374E+03	-7.7252E+04	-2.3626E+06
1.020	-4.5035E+00	-1.7049E+01	-1.6731E+02	-2.7792E+03	-6.4454E+04	-1.9167E+06
1.025	-4.2775E+00	-1.5809E+01	-1.5090E+02	-2.4431E+03	-5.5267E+04	-1.6038E+06
1.030	-4.0929E+00	-1.4812E+01	-1.3792E+02	-2.1823E+03	-4.8278E+04	-1.3705E+06
1.035	-3.9370E+00	-1.3982E+01	-1.2728E+02	-1.9722E+03	-4.2750E+04	-1.1894E+06
1.040	-3.8020E+00	-1.3273E+01	-1.1833E+02	-1.7984E+03	-3.8252E+04	-1.0446E+06
1.044	-3.7057E+00	-1.2773E+01	-1.1211E+02	-1.6791E+03	-3.5211E+04	-9.4809E+05
1.077	-3.1418E+00	-9.9663E+00	-7.8589E+01	-1.0652E+03	-2.0260E+04	-4.9527E+05
2.000	-6.7432E-01	-1.5998E+00	-4.4739E+00	-2.2387E+01	-1.6359E+02	-1.5564E+03
2.250	-4.7854E-01	-1.2795E+00	-3.1297E+00	-1.3397E+01	-8.4847E+01	-7.0295E+02
2.500	-3.2283E-01	-1.0530E+00	-2.3253E+00	-8.6526E+00	-4.8254E+01	-3.5403E+02

2.750	-1.9520E-01	-8.8157E-01	-1.8079E+00	-5.9238E+00	-2.9425E+01	-1.9356E+02
3.000	-8.8757E-02	-7.4478E-01	-1.4560E+00	-4.2494E+00	-1.8961E+01	-1.1290E+02
3.250	8.0709E-04	-6.3098E-01	-1.2054E+00	-3.1681E+00	-1.2782E+01	-6.9409E+01
3.500	7.6327E-02	-5.3320E-01	-1.0198E+00	-2.4405E+00	-8.9503E+00	-4.4586E+01
3.750	1.3977E-01	-4.4705E-01	-8.7730E-01	-1.9336E+00	-6.4744E+00	-2.9730E+01
4.000	1.9258E-01	-3.6974E-01	-7.6425E-01	-1.5698E+00	-4.8184E+00	-2.0473E+01
4.250	2.3585E-01	-2.9943E-01	-6.7177E-01	-1.3019E+00	-3.6775E+00	-1.4503E+01
4.500	2.7044E-01	-2.3494E-01	-5.9392E-01	-1.0999E+00	-2.8708E+00	-1.0535E+01
4.750	2.9708E-01	-1.7548E-01	-5.2665E-01	-9.4423E-01	-2.2874E+00	-7.8269E+00
5.000	3.1642E-01	-1.2056E-01	-4.6714E-01	-8.2177E-01	-1.8568E+00	-5.9343E+00
6.000	3.3163E-01	5.8386E-02	-2.7474E-01	-5.1848E-01	-9.4306E-01	-2.3294E+00
7.000	2.7175E-01	1.7495E-01	-1.2120E-01	-3.4838E-01	-5.7875E-01	-1.1430E+00
8.000	1.6823E-01	2.3014E-01	5.8141E-03	-2.2261E-01	-3.9666E-01	-6.6721E-01
9.000	5.1104E-02	2.2880E-01	1.0301E-01	-1.1403E-01	-2.8084E-01	-4.4313E-01
10.000	-5.3681E-02	1.8209E-01	1.6426E-01	-1.7641E-02	-1.8905E-01	-3.1844E-01

$\beta = 0.75$

u\mode	$\mathrm{Im}(R^{(0)})$	$\mathrm{Im}(R^{(1)})$	$\mathrm{Im}(R^{(2)})$	$\mathrm{Im}(R^{(3)})$	$\mathrm{Im}(R^{(4)})$	$\mathrm{Im}(R^{(5)})$
1.005	-3.8719E+00	-1.1640E+01	-8.3603E+01	-1.0303E+03	-1.7728E+04	-3.9062E+05
1.010	-3.3932E+00	-9.7944E+00	-6.6534E+01	-7.8068E+02	-1.2823E+04	-2.7006E+05
1.015	-3.1125E+00	-8.7353E+00	-5.6974E+01	-6.4466E+02	-1.0228E+04	-2.0825E+05
1.020	-2.9129E+00	-7.9969E+00	-5.0445E+01	-5.5388E+02	-8.5385E+03	-1.6902E+05
1.025	-2.7579E+00	-7.4333E+00	-4.5554E+01	-4.8726E+02	-7.3255E+03	-1.4149E+05
1.030	-2.6310E+00	-6.9798E+00	-4.1685E+01	-4.3554E+02	-6.4022E+03	-1.2095E+05
1.035	-2.5235E+00	-6.6020E+00	-3.8513E+01	-3.9387E+02	-5.6717E+03	-1.0500E+05
1.040	-2.4304E+00	-6.2793E+00	-3.5843E+01	-3.5937E+02	-5.0772E+03	-9.2252E+04
1.044	-2.3638E+00	-6.0517E+00	-3.3986E+01	-3.3570E+02	-4.6751E+03	-8.3754E+04
1.077	-1.9715E+00	-4.7712E+00	-2.3972E+01	-2.1371E+02	-2.6967E+03	-4.3838E+04
2.000	-1.5049E-01	-8.1728E-01	-1.6284E+00	-4.9957E+00	-2.3369E+01	-1.4535E+02
2.250	-1.5197E-03	-6.2739E-01	-1.2000E+00	-3.1186E+00	-1.2450E+01	-6.6952E+01
2.500	1.1122E-01	-4.7970E-01	-9.3380E-01	-2.1143E+00	-7.3041E+00	-3.4481E+01
2.750	1.9602E-01	-3.5758E-01	-7.5277E-01	-1.5278E+00	-4.6162E+00	-1.9333E+01
3.000	2.5788E-01	-2.5281E-01	-6.1974E-01	-1.1612E+00	-3.0987E+00	-1.1600E+01
3.250	3.0016E-01	-1.6113E-01	-5.1520E-01	-9.1859E-01	-2.1879E+00	-7.3612E+00
3.500	3.2544E-01	-8.0359E-02	-4.2826E-01	-7.4977E-01	-1.6131E+00	-4.8988E+00
3.750	3.3585E-01	-9.3805E-03	-3.5264E-01	-6.2651E-01	-1.2349E+00	-3.3977E+00
4.000	3.3331E-01	5.2310E-02	-2.8467E-01	-5.3220E-01	-9.7664E-01	-2.4444E+00
4.250	3.1965E-01	1.0492E-01	-2.2222E-01	-4.5662E-01	-7.9434E-01	-1.8174E+00
4.500	2.9662E-01	1.4853E-01	-1.6418E-01	-3.9334E-01	-6.6151E-01	-1.3920E+00
4.750	2.6596E-01	1.8321E-01	-1.1000E-01	-3.3817E-01	-5.6160E-01	-1.0955E+00
5.000	2.2940E-01	2.0906E-01	-5.9565E-02	-2.8843E-01	-4.8396E-01	-8.8369E-01
6.000	5.6554E-02	2.3023E-01	1.0060E-01	-1.1778E-01	-2.8491E-01	-4.5477E-01
7.000	-9.2157E-02	1.4894E-01	1.7985E-01	2.2636E-02	-1.4949E-01	-2.7941E-01
8.000	-1.5930E-01	2.3147E-02	1.7179E-01	1.1994E-01	-3.1398E-02	-1.6711E-01
9.000	-1.3468E-01	-8.4660E-02	9.6640E-02	1.5607E-01	6.5587E-02	-6.8833E-02
10.000	-4.9541E-02	-1.3130E-01	-3.9966E-03	1.2856E-01	1.2419E-01	2.0104E-02

β = 1.0

u\mode	$Im(R^{(0)})$	$Im(R^{(1)})$	$Im(R^{(2)})$	$Im(R^{(3)})$	$Im(R^{(4)})$	$Im(R^{(5)})$
1.005	**-2.8378E+00**	-6.9118E+00	**-3.5926E+01**	-3.2875E+02	-4.2297E+03	-6.9793E+04
1.010	-2.4683E+00	-5.8434E+00	-2.8681E+01	-2.4954E+02	-3.0631E+03	-4.8299E+04
1.015	-2.2508E+00	-5.2296E+00	-2.4619E+01	-2.0634E+02	-2.4457E+03	-3.7273E+04
1.020	**-2.0958E+00**	-4.8011E+00	**-2.1843E+01**	-1.7750E+02	-2.0434E+03	-3.0272E+04
1.025	-1.9750E+00	-4.4737E+00	-1.9763E+01	-1.5631E+02	-1.7544E+03	-2.5355E+04
1.030	-1.8759E+00	-4.2100E+00	-1.8115E+01	-1.3986E+02	-1.5344E+03	-2.1687E+04
1.035	-1.7918E+00	-3.9900E+00	-1.6764E+01	-1.2659E+02	-1.3602E+03	-1.8837E+04
1.040	-1.7187E+00	-3.8019E+00	-1.5627E+01	-1.1561E+02	-1.2184E+03	-1.6557E+04
1.044	**-1.6663E+00**	-3.6692E+00	**-1.4835E+01**	-1.0807E+02	-1.1224E+03	-1.5038E+04
1.077	**-1.3557E+00**	-2.9198E+00	**-1.0558E+01**	-6.9145E+01	-6.4972E+02	-7.8925E+03
2.000	1.3356E-01	-4.3844E-01	-8.7581E-01	-1.9004E+00	-6.2538E+00	-2.8271E+01
2.250	2.3678E-01	-2.8074E-01	-6.6027E-01	-1.2606E+00	-3.4764E+00	-1.3411E+01
2.500	2.9976E-01	-1.5231E-01	-5.1125E-01	-9.0922E-01	-2.1445E+00	-7.1474E+00
2.750	3.3052E-01	-4.5418E-02	-3.9591E-01	-6.9540E-01	-1.4359E+00	-4.1701E+00
3.000	3.3491E-01	4.3072E-02	-2.9917E-01	-5.5273E-01	-1.0275E+00	-2.6200E+00
3.250	3.1788E-01	1.1439E-01	-2.1389E-01	-4.4884E-01	-7.7597E-01	-1.7528E+00
3.500	2.8401E-01	1.6905E-01	-1.3688E-01	-3.6673E-01	-6.1137E-01	-1.2382E+00
3.750	2.3765E-01	2.0743E-01	-6.7025E-02	-2.9711E-01	-4.9709E-01	-9.1724E-01
4.000	1.8300E-01	2.3010E-01	-4.3206E-03	-2.3487E-01	-4.1271E-01	-7.0806E-01
4.250	1.2402E-01	2.3793E-01	5.0675E-02	-1.7728E-01	-3.4631E-01	-5.6591E-01
4.500	6.4411E-02	2.3221E-01	9.7195E-02	-1.2310E-01	-2.9072E-01	-4.6512E-01
4.750	7.4644E-03	2.1461E-01	1.3449E-01	-7.2038E-02	-2.4156E-01	-3.9035E-01
5.000	-4.3993E-02	1.8713E-01	1.6198E-01	-2.4349E-02	-1.9623E-01	-3.3206E-01
6.000	-1.5857E-01	2.5760E-02	1.7317E-01	1.1849E-01	-3.4078E-02	-1.6993E-01
7.000	-1.1292E-01	-1.0738E-01	6.5608E-02	1.5401E-01	8.8974E-02	-3.9263E-02
8.000	1.2954E-02	-1.2298E-01	-6.0967E-02	8.4923E-02	1.3543E-01	6.7774E-02
9.000	9.9834E-02	-3.7101E-02	-1.1301E-01	-2.5410E-02	9.3193E-02	1.1880E-01
10.000	8.6000E-02	6.1441E-02	-6.7083E-02	-9.4775E-02	6.7449E-04	9.5493E-02

Table A-7. Values of real and imaginary parts of $R^{(n)}_u(\beta, u)$ for six modes (four values of β values and numerous values of u).

β = 0.25

u\mode	$Re(R_u^{(0)})$	$Re(R_u^{(1)})$	$Re(R_u^{(2)})$	$Re(R_u^{(3)})$	$Re(R_u^{(4)})$	$Re(R_u^{(5)})$
1.005	-2.0893E-02	8.1967E-02	8.3297E-03	3.5990E-04	-3.2266E-04	3.6870E-02
1.010	-2.0995E-02	8.1951E-02	8.3704E-03	3.6436E-04	-3.2414E-04	3.6859E-02
1.015	-2.1098E-02	8.1936E-02	8.4111E-03	3.6885E-04	-3.2557E-04	3.6851E-02
1.020	-2.1201E-02	8.1920E-02	8.4518E-03	3.7335E-04	-3.2699E-04	3.6843E-02
1.025	-2.1303E-02	8.1904E-02	8.4924E-03	3.7788E-04	-3.2841E-04	3.6835E-02
1.030	-2.1406E-02	8.1888E-02	8.5331E-03	3.8243E-04	-3.2982E-04	3.6828E-02
1.035	-2.1508E-02	8.1872E-02	8.5737E-03	3.8700E-04	-3.3122E-04	3.6821E-02
1.040	-2.1611E-02	8.1856E-02	8.6143E-03	3.9159E-04	-3.3262E-04	3.6813E-02
1.044	-2.1693E-02	8.1843E-02	8.6468E-03	3.9528E-04	-3.3374E-04	3.6807E-02
1.077	-2.2369E-02	8.1734E-02	8.9145E-03	4.2624E-04	-3.4290E-04	3.6758E-02

2.000	-4.0805E-02	7.7374E-02	1.6136E-02	1.6613E-03	-5.3280E-04	3.4797E-02
2.250	-4.5600E-02	7.5766E-02	1.7979E-02	2.1129E-03	-5.5634E-04	3.4076E-02
2.500	-5.0289E-02	7.3984E-02	1.9762E-02	2.6108E-03	-5.6516E-04	3.3278E-02
2.750	-5.4861E-02	7.2033E-02	2.1479E-02	3.1530E-03	-5.5791E-04	3.2406E-02
3.000	-5.9305E-02	6.9919E-02	2.3123E-02	3.7371E-03	-5.3330E-04	3.1463E-02
3.250	-6.3613E-02	6.7647E-02	2.4691E-02	4.3604E-03	-4.9014E-04	3.0451E-02
3.500	-6.7773E-02	6.5224E-02	2.6175E-02	5.0203E-03	-4.2733E-04	2.9375E-02
3.750	-7.1777E-02	6.2655E-02	2.7572E-02	5.7138E-03	-3.4387E-04	2.8237E-02
4.000	-7.5616E-02	5.9950E-02	2.8876E-02	6.4379E-03	-2.3888E-04	2.7042E-02
4.250	-7.9282E-02	5.7113E-02	3.0083E-02	7.1893E-03	-1.1157E-04	2.5792E-02
4.500	-8.2766E-02	5.4155E-02	3.1189E-02	7.9646E-03	3.8710E-05	2.4493E-02
4.750	-8.6062E-02	5.1082E-02	3.2191E-02	8.7605E-03	2.1250E-04	2.3149E-02
5.000	-8.9162E-02	4.7904E-02	3.3083E-02	9.5732E-03	4.1021E-04	2.1763E-02
6.000	-9.9487E-02	3.4310E-02	3.5516E-02	1.2918E-02	1.4439E-03	1.5900E-02
7.000	-1.0628E-01	1.9758E-02	3.6034E-02	1.6227E-02	2.8590E-03	9.7531E-03
8.000	-1.0937E-01	4.8731E-03	3.4597E-02	1.9248E-02	4.6176E-03	3.6326E-03
9.000	-1.0876E-01	-9.7133E-03	3.1266E-02	2.1738E-02	6.6482E-03	-2.1567E-03
10.000	-1.0460E-01	-2.3395E-02	2.6197E-02	2.3478E-02	8.8493E-03	-7.3338E-03

$\beta = 0.5$

u\mode	Re($R_u^{(0)}$)	Re($R_u^{(1)}$)	Re($R_u^{(2)}$)	Re($R_u^{(3)}$)	Re($R_u^{(4)}$)	Re($R_u^{(5)}$)
1.005	**-8.3043E-02**	**1.5582E-01**	**3.2776E-02**	**2.8612E-03**	8.0216E-05	4.1917E-03
1.010	-8.3435E-02	1.5570E-01	3.2927E-02	2.8961E-03	8.3273E-05	4.1885E-03
1.015	-8.3826E-02	1.5558E-01	3.3078E-02	2.9311E-03	8.6369E-05	4.1853E-03
1.020	**-8.4218E-02**	**1.5545E-01**	**3.3228E-02**	**2.9662E-03**	8.9505E-05	4.1821E-03
1.025	-8.4609E-02	1.5533E-01	3.3379E-02	3.0015E-03	9.2680E-05	4.1789E-03
1.030	-8.5000E-02	1.5520E-01	3.3529E-02	3.0370E-03	9.5894E-05	4.1757E-03
1.035	-8.5390E-02	1.5508E-01	3.3679E-02	3.0726E-03	9.9148E-05	4.1725E-03
1.040	-8.5781E-02	1.5495E-01	3.3829E-02	3.1084E-03	1.0244E-04	4.1693E-03
1.044	**-8.6092E-02**	**1.5485E-01**	**3.3949E-02**	**3.1371E-03**	1.0510E-04	4.1667E-03
1.077	**-8.8658E-02**	**1.5400E-01**	**3.4934E-02**	**3.3779E-03**	1.2805E-04	4.1452E-03
2.000	-1.5320E-01	1.2091E-01	5.8409E-02	1.2493E-02	1.6443E-03	3.4279E-03
2.250	-1.6771E-01	1.0925E-01	6.3100E-02	1.5584E-02	2.4019E-03	3.2344E-03
2.500	-1.8069E-01	9.6679E-02	6.6947E-02	1.8842E-02	3.3267E-03	3.0634E-03
2.750	-1.9204E-01	8.3325E-02	6.9897E-02	2.2209E-02	4.4228E-03	2.9275E-03
3.000	-2.0167E-01	6.9336E-02	7.1909E-02	2.5622E-02	5.6904E-03	2.8394E-03
3.250	-2.0951E-01	5.4862E-02	7.2954E-02	2.9019E-02	7.1255E-03	2.8117E-03
3.500	-2.1551E-01	4.0061E-02	7.3016E-02	3.2337E-02	8.7201E-03	2.8565E-03
3.750	-2.1964E-01	2.5092E-02	7.2089E-02	3.5510E-02	1.0462E-02	2.9851E-03
4.000	-2.2189E-01	1.0112E-02	7.0183E-02	3.8478E-02	1.2336E-02	3.2074E-03
4.250	-2.2226E-01	-4.7176E-03	6.7319E-02	4.1180E-02	1.4321E-02	3.5323E-03
4.500	-2.2078E-01	-1.9243E-02	6.3531E-02	4.3557E-02	1.6394E-02	3.9670E-03
4.750	-2.1751E-01	-3.3315E-02	5.8864E-02	4.5558E-02	1.8528E-02	4.5165E-03
5.000	-2.1250E-01	-4.6788E-02	5.3374E-02	4.7132E-02	2.0695E-02	5.1841E-03
6.000	-1.7704E-01	-9.2146E-02	2.4666E-02	4.8402E-02	2.9018E-02	9.0131E-03
7.000	-1.2315E-01	-1.1916E-01	-1.0392E-02	4.0634E-02	3.5076E-02	1.4342E-02
8.000	-6.0979E-02	-1.2464E-01	-4.5034E-02	2.4158E-02	3.6714E-02	2.0189E-02

9.000	-1.3240E-03	-1.0938E-01	-7.2657E-02	1.2888E-03	3.2426E-02	2.5100E-02
10.000	4.6396E-02	-7.7898E-02	-8.8148E-02	-2.4105E-02	2.1822E-02	2.7452E-02

β = 0.75

u\mode	Re($R_u^{(0)}$)	Re($R_u^{(1)}$)	Re($R_u^{(2)}$)	Re($R_u^{(3)}$)	Re($R_u^{(4)}$)	Re($R_u^{(5)}$)
1.005	-1.8493E-01	2.1388E-01	7.1707E-02	9.5249E-03	7.4942E-04	1.0883E-03
1.010	-1.8574E-01	2.1347E-01	7.2004E-02	9.6377E-03	7.6620E-04	1.0878E-03
1.015	-1.8655E-01	2.1307E-01	7.2300E-02	9.7510E-03	7.8317E-04	1.0873E-03
1.020	-1.8737E-01	2.1266E-01	7.2595E-02	9.8647E-03	8.0032E-04	1.0869E-03
1.025	-1.8818E-01	2.1225E-01	7.2890E-02	9.9788E-03	8.1766E-04	1.0864E-03
1.030	-1.8898E-01	2.1184E-01	7.3184E-02	1.0093E-02	8.3519E-04	1.0860E-03
1.035	-1.8979E-01	2.1143E-01	7.3476E-02	1.0208E-02	8.5290E-04	1.0856E-03
1.040	-1.9059E-01	2.1102E-01	7.3768E-02	1.0324E-02	8.7080E-04	1.0853E-03
1.044	-1.9124E-01	2.1068E-01	7.4001E-02	1.0416E-02	8.8525E-04	1.0850E-03
1.077	-1.9650E-01	2.0790E-01	7.5901E-02	1.1191E-02	1.0091E-03	1.0833E-03
2.000	-3.0942E-01	1.0583E-01	1.1004E-01	3.7892E-02	8.3082E-03	1.8727E-03
2.250	-3.2656E-01	7.2745E-02	1.1197E-01	4.5678E-02	1.1633E-02	2.5469E-03
2.500	-3.3733E-01	3.8806E-02	1.1054E-01	5.3093E-02	1.5487E-02	3.5042E-03
2.750	-3.4164E-01	4.8251E-03	1.0577E-01	5.9815E-02	1.9791E-02	4.7805E-03
3.000	-3.3956E-01	-2.8393E-02	9.7770E-02	6.5538E-02	2.4443E-02	6.4013E-03
3.250	-3.3131E-01	-6.0077E-02	8.6771E-02	6.9984E-02	2.9309E-02	8.3794E-03
3.500	-3.1723E-01	-8.9504E-02	7.3068E-02	7.2908E-02	3.4238E-02	1.0712E-02
3.750	-2.9783E-01	-1.1602E-01	5.7038E-02	7.4112E-02	3.9059E-02	1.3381E-02
4.000	-2.7370E-01	-1.3905E-01	3.9124E-02	7.3448E-02	4.3591E-02	1.6350E-02
4.250	-2.4556E-01	-1.5814E-01	1.9821E-02	7.0827E-02	4.7648E-02	1.9565E-02
4.500	-2.1417E-01	-1.7292E-01	-3.3967E-04	6.6221E-02	5.1046E-02	2.2955E-02
4.750	-1.8039E-01	-1.8315E-01	-2.0802E-02	5.9664E-02	5.3610E-02	2.6433E-02
5.000	-1.4507E-01	-1.8870E-01	-4.1007E-02	5.1253E-02	5.5179E-02	2.9897E-02
6.000	-5.9636E-03	-1.6615E-01	-1.0872E-01	3.0419E-03	4.9195E-02	4.1275E-02
7.000	9.3959E-02	-8.9174E-02	-1.3622E-01	-5.3763E-02	2.2234E-02	4.2951E-02
8.000	1.2667E-01	4.9579E-03	-1.1383E-01	-9.6138E-02	-1.9318E-02	2.9561E-02
9.000	9.4356E-02	7.6547E-02	-5.3506E-02	-1.0612E-01	-6.0936E-02	1.5803E-03
10.000	2.4772E-02	1.0019E-01	1.7478E-02	-7.9302E-02	-8.5983E-02	-3.3439E-02

β = 1.0

u\mode	Re($R_u^{(0)}$)	Re($R_u^{(1)}$)	Re($R_u^{(2)}$)	Re($R_u^{(3)}$)	Re($R_u^{(4)}$)	Re($R_u^{(5)}$)
1.005	**-3.2421E-01**	**2.4925E-01**	**1.2241E-01**	**2.2138E-02**	2.4177E-03	5.1795E-04
1.010	-3.2549E-01	2.4833E-01	1.2284E-01	2.2390E-02	2.4695E-03	5.2298E-04
1.015	-3.2677E-01	2.4741E-01	1.2326E-01	2.2642E-02	2.5219E-03	5.2813E-04
1.020	**-3.2804E-01**	**2.4648E-01**	**1.2368E-01**	**2.2896E-02**	2.5748E-03	5.3340E-04
1.025	-3.2931E-01	2.4556E-01	1.2409E-01	2.3150E-02	2.6282E-03	5.3878E-04
1.030	-3.3057E-01	2.4463E-01	1.2451E-01	2.3405E-02	2.6822E-03	5.4428E-04
1.035	-3.3183E-01	2.4369E-01	1.2492E-01	2.3660E-02	2.7367E-03	5.4990E-04
1.040	-3.3308E-01	2.4275E-01	1.2533E-01	2.3916E-02	2.7918E-03	5.5564E-04
1.044	**-3.3408E-01**	**2.4200E-01**	**1.2565E-01**	**2.4122E-02**	2.8362E-03	5.6032E-04
1.077	**-3.4220E-01**	**2.3571E-01**	**1.2827E-01**	**2.5834E-02**	3.2161E-03	6.0201E-04
2.000	-4.7024E-01	2.3289E-02	1.4814E-01	7.6798E-02	2.3640E-02	5.3242E-03
2.250	-4.6902E-01	-3.6937E-02	1.3491E-01	8.7756E-02	3.1898E-02	8.2633E-03

2.500	-4.5280E-01	-9.3522E-02	1.1444E-01	9.5669E-02	4.0706E-02	1.2049E-02
2.750	-4.2279E-01	-1.4419E-01	8.7727E-02	9.9774E-02	4.9565E-02	1.6660E-02
3.000	-3.8075E-01	-1.8699E-01	5.6071E-02	9.9510E-02	5.7913E-02	2.2007E-02
3.250	-3.2888E-01	-2.2036E-01	2.1025E-02	9.4549E-02	6.5157E-02	2.7923E-02
3.500	-2.6972E-01	-2.4320E-01	-1.5697E-02	8.4826E-02	7.0710E-02	3.4174E-02
3.750	-2.0602E-01	-2.5495E-01	-5.2311E-02	7.0543E-02	7.4032E-02	4.0459E-02
4.000	-1.4063E-01	-2.5553E-01	-8.7051E-02	5.2163E-02	7.4665E-02	4.6426E-02
4.250	-7.6343E-02	-2.4540E-01	-1.1826E-01	3.0391E-02	7.2267E-02	5.1691E-02
4.500	-1.5773E-02	-2.2551E-01	-1.4447E-01	6.1372E-03	6.6642E-02	5.5859E-02
4.750	3.8745E-02	-1.9719E-01	-1.6448E-01	-1.9529E-02	5.7759E-02	5.8543E-02
5.000	8.5268E-02	-1.6217E-01	-1.7742E-01	-4.5436E-02	4.5761E-02	5.9398E-02
6.000	1.6964E-01	4.3711E-03	-1.5383E-01	-1.2796E-01	-2.4588E-02	4.0159E-02
7.000	1.0082E-01	1.2020E-01	-4.1252E-02	-1.3572E-01	-9.5393E-02	-1.2739E-02
8.000	-2.8809E-02	1.1751E-01	7.3371E-02	-6.2836E-02	-1.1724E-01	-7.2828E-02
9.000	-1.0584E-01	2.4694E-02	1.1066E-01	3.7316E-02	-7.1704E-02	-1.0077E-01
10.000	-8.1093E-02	-6.9104E-02	5.7242E-02	9.3694E-02	1.1350E-02	-7.3967E-02

$\beta = 0.25$

u\mode	$Im(R_u^{(0)})$	$Im(R_u^{(1)})$	$Im(R_u^{(2)})$	$Im(R_u^{(3)})$	$Im(R_u^{(4)})$	$Im(R_u^{(5)})$
1.005	4.0068E+02	4.6866E+03	1.3503E+05	6.0216E+06	3.6041E+08	2.6991E+10
1.010	1.9996E+02	2.2940E+03	6.4440E+04	2.7910E+06	1.6183E+08	1.1721E+10
1.015	1.3305E+02	1.5006E+03	4.1255E+04	1.7440E+06	9.8528E+07	6.9448E+09
1.020	9.9590E+01	1.1058E+03	2.9825E+04	1.2341E+06	6.8160E+07	4.6926E+09
1.025	7.9514E+01	8.7015E+02	2.3060E+04	9.3589E+05	5.0644E+07	3.4139E+09
1.030	6.6130E+01	7.1381E+02	1.8610E+04	7.4192E+05	3.9402E+07	2.6052E+09
1.035	5.6570E+01	6.0269E+02	1.5474E+04	6.0668E+05	3.1662E+07	2.0563E+09
1.040	4.9400E+01	5.1975E+02	1.3152E+04	5.0760E+05	2.6061E+07	1.6644E+09
1.044	4.4838E+01	4.6720E+02	1.1692E+04	4.4586E+05	2.2608E+07	1.4255E+09
1.077	2.5286E+01	2.4536E+02	5.6696E+03	1.9859E+05	9.2235E+06	5.3182E+08
2.000	1.4642E+00	5.6977E+00	5.4653E+01	7.7736E+02	1.4524E+04	3.3527E+05
2.250	1.1102E+00	3.7086E+00	3.1153E+01	3.8801E+02	6.3435E+03	1.2812E+05
2.500	8.8318E-01	2.5749E+00	1.9240E+01	2.1348E+02	3.1068E+03	5.5854E+04
2.750	7.2688E-01	1.8750E+00	1.2602E+01	1.2620E+02	1.6562E+03	2.6850E+04
3.000	6.1363E-01	1.4170E+00	8.6367E+00	7.8861E+01	9.4292E+02	1.3926E+04
3.250	5.2826E-01	1.1037E+00	6.1371E+00	5.1524E+01	5.6592E+02	7.6776E+03
3.500	4.6184E-01	8.8185E-01	4.4923E+00	3.4917E+01	3.5472E+02	4.4509E+03
3.750	4.0877E-01	7.2014E-01	3.3711E+00	2.4399E+01	2.3060E+02	2.6916E+03
4.000	3.6540E-01	5.9947E-01	2.5841E+00	1.7501E+01	1.5465E+02	1.6875E+03
4.250	3.2926E-01	5.0761E-01	2.0176E+00	1.2840E+01	1.0654E+02	1.0914E+03
4.500	2.9860E-01	4.3647E-01	1.6010E+00	9.6076E+00	7.5142E+01	7.2542E+02
4.750	2.7220E-01	3.8055E-01	1.2888E+00	7.3150E+00	5.4109E+01	4.9390E+02
5.000	2.4913E-01	3.3599E-01	1.0510E+00	5.6562E+00	3.9689E+01	3.4355E+02
6.000	1.7890E-01	2.2656E-01	5.1860E-01	2.2925E+00	1.3353E+01	9.5766E+01
7.000	1.2895E-01	1.7259E-01	2.9598E-01	1.0831E+00	5.3953E+00	3.3039E+01
8.000	8.9592E-02	1.4145E-01	1.9127E-01	5.7297E-01	2.4895E+00	1.3317E+01
9.000	5.6778E-02	1.2019E-01	1.3805E-01	3.3190E-01	1.2706E+00	6.0487E+00
10.000	2.8757E-02	1.0313E-01	1.0941E-01	2.0831E-01	7.0218E-01	3.0215E+00

$\beta = 0.5$

u\mode	$Im(R_u^{(0)})$	$Im(R_u^{(1)})$	$Im(R_u^{(2)})$	$Im(R_u^{(3)})$	$Im(R_u^{(4)})$	$Im(R_u^{(5)})$
1.005	2.0284E+02	1.1856E+03	1.6911E+04	3.7671E+05	1.1271E+07	4.2198E+08
1.010	1.0140E+02	5.8059E+02	8.0737E+03	1.7467E+05	5.0625E+06	1.8329E+08
1.015	6.7578E+01	3.7991E+02	5.1708E+03	1.0918E+05	3.0830E+06	1.0863E+08
1.020	5.0660E+01	2.8006E+02	3.7393E+03	7.7281E+04	2.1333E+06	7.3416E+07
1.025	4.0505E+01	2.2043E+02	2.8920E+03	5.8621E+04	1.5854E+06	5.3420E+07
1.030	3.3733E+01	1.8087E+02	2.3346E+03	4.6482E+04	1.2337E+06	4.0774E+07
1.035	2.8895E+01	1.5275E+02	1.9417E+03	3.8017E+04	9.9157E+05	3.2188E+07
1.040	2.5266E+01	1.3176E+02	1.6507E+03	3.1816E+04	8.1630E+05	2.6057E+07
1.044	2.2955E+01	1.1846E+02	1.4677E+03	2.7951E+04	7.0823E+05	2.2320E+07
1.077	1.3048E+01	6.2290E+01	7.1268E+02	1.2464E+04	2.8923E+05	8.3340E+06
2.000	8.8812E-01	1.5552E+00	7.0571E+00	5.0165E+01	4.6674E+02	5.3668E+03
2.250	6.9248E-01	1.0583E+00	4.0585E+00	2.5270E+01	2.0559E+02	2.0662E+03
2.500	5.6085E-01	7.7790E-01	2.5336E+00	1.4041E+01	1.0164E+02	9.0823E+02
2.750	4.6462E-01	6.0652E-01	1.6823E+00	8.3873E+00	5.4741E+01	4.4060E+02
3.000	3.8975E-01	4.9533E-01	1.1738E+00	5.2996E+00	3.1512E+01	2.3081E+02
3.250	3.2862E-01	4.1956E-01	8.5409E-01	3.5031E+00	1.9138E+01	1.2862E+02
3.500	2.7683E-01	3.6558E-01	6.4499E-01	2.4035E+00	1.2148E+01	7.5436E+01
3.750	2.3169E-01	3.2544E-01	5.0390E-01	1.7020E+00	8.0027E+00	4.6188E+01
4.000	1.9151E-01	2.9426E-01	4.0631E-01	1.2388E+00	5.4416E+00	2.9342E+01
4.250	1.5519E-01	2.6895E-01	3.3744E-01	9.2403E-01	3.8031E+00	1.9245E+01
4.500	1.2202E-01	2.4749E-01	2.8801E-01	7.0484E-01	2.7226E+00	1.2980E+01
4.750	9.1556E-02	2.2851E-01	2.5201E-01	5.4909E-01	1.9909E+00	8.9748E+00
5.000	6.3486E-02	2.1111E-01	2.2540E-01	4.3652E-01	1.4838E+00	6.3437E+00
6.000	-2.7629E-02	1.4760E-01	1.6850E-01	2.1272E-01	5.3720E-01	1.8959E+00
7.000	-8.6885E-02	8.5557E-02	1.4015E-01	1.4040E-01	2.4306E-01	7.0621E-01
8.000	-1.1509E-01	2.5598E-02	1.1315E-01	1.1515E-01	1.3827E-01	3.1092E-01
9.000	-1.1478E-01	-2.6403E-02	8.0133E-02	1.0265E-01	9.9788E-02	1.5992E-01
10.000	-9.1600E-02	-6.4204E-02	4.1770E-02	8.9402E-02	8.6076E-02	9.8508E-02

$\beta = 0.75$

u\mode	$Im(R_u^{(0)})$	$Im(R_u^{(1)})$	$Im(R_u^{(2)})$	$Im(R_u^{(3)})$	$Im(R_u^{(4)})$	$Im(R_u^{(5)})$
1.005	1.3797E+02	5.3751E+02	5.0281E+03	7.4534E+04	1.4860E+06	3.7082E+07
1.010	6.9163E+01	2.6343E+02	2.4023E+03	3.4581E+04	6.6782E+05	1.6115E+07
1.015	4.6205E+01	1.7250E+02	1.5395E+03	2.1626E+04	4.0688E+05	9.5543E+06
1.020	3.4717E+01	1.2724E+02	1.1139E+03	1.5315E+04	2.8165E+05	6.4593E+06
1.025	2.7818E+01	1.0021E+02	8.6189E+02	1.1622E+04	2.0939E+05	4.7015E+06
1.030	2.3216E+01	8.2277E+01	6.9608E+02	9.2192E+03	1.6300E+05	3.5895E+06
1.035	1.9926E+01	6.9523E+01	5.7915E+02	7.5430E+03	1.3104E+05	2.8344E+06
1.040	1.7456E+01	6.0001E+01	4.9256E+02	6.3147E+03	1.0791E+05	2.2950E+06
1.044	1.5884E+01	5.3967E+01	4.3808E+02	5.5490E+03	9.3649E+04	1.9663E+06
1.077	9.1317E+00	2.8470E+01	2.1317E+02	2.4793E+03	3.8308E+04	7.3525E+05
2.000	6.8543E-01	8.8180E-01	2.2211E+00	1.0415E+01	6.4353E+01	4.9051E+02
2.250	5.1624E-01	6.5977E-01	1.3164E+00	5.3153E+00	2.8730E+01	1.9117E+02
2.500	3.9098E-01	5.3219E-01	8.6075E-01	2.9957E+00	1.4410E+01	8.5184E+01
2.750	2.9066E-01	4.4998E-01	6.1095E-01	1.8196E+00	7.8811E+00	4.1942E+01

3.000	2.0636E-01	3.9092E-01	4.6576E-01	1.1742E+00	4.6103E+00	2.2326E+01
3.250	1.3359E-01	3.4389E-01	3.7760E-01	7.9860E-01	2.8473E+00	1.2656E+01
3.500	7.0041E-02	3.0302E-01	3.2198E-01	5.7016E-01	1.8396E+00	7.5571E+00
3.750	1.4510E-02	2.6513E-01	2.8536E-01	4.2667E-01	1.2355E+00	4.7146E+00
4.000	-3.3594E-02	2.2852E-01	2.5981E-01	3.3447E-01	8.5890E-01	3.0536E+00
4.250	-7.4555E-02	1.9241E-01	2.4046E-01	2.7432E-01	6.1663E-01	2.0431E+00
4.500	-1.0851E-01	1.5652E-01	2.2426E-01	2.3469E-01	4.5682E-01	1.4068E+00
4.750	-1.3557E-01	1.2095E-01	2.0924E-01	2.0838E-01	3.4941E-01	9.9387E-01
5.000	-1.5584E-01	8.6009E-02	1.9415E-01	1.9069E-01	2.7628E-01	7.1903E-01
6.000	-1.7396E-01	-3.8115E-02	1.2263E-01	1.5557E-01	1.5208E-01	2.4747E-01
7.000	-1.1368E-01	-1.1443E-01	3.4553E-02	1.2242E-01	1.2498E-01	1.2948E-01
8.000	-1.9010E-02	-1.2631E-01	-4.7196E-02	6.8851E-02	1.1006E-01	1.0216E-01
9.000	6.2668E-02	-8.1849E-02	-9.5857E-02	2.9089E-03	8.0729E-02	9.4810E-02
10.000	9.8505E-02	-9.8836E-03	-9.7431E-02	-5.4657E-02	3.4251E-02	8.0796E-02

$\beta = 1.0$

u\mode	$\mathrm{Im}(R_u^{(0)})$	$\mathrm{Im}(R_u^{(1)})$	$\mathrm{Im}(R_u^{(2)})$	$\mathrm{Im}(R_u^{(3)})$	$\mathrm{Im}(R_u^{(4)})$	$\mathrm{Im}(R_u^{(5)})$
1.005	1.0633E+02	3.1089E+02	2.1332E+03	2.3640E+04	3.5324E+05	6.6089E+06
1.010	5.3490E+01	1.5258E+02	1.0202E+03	1.0978E+04	1.5887E+05	2.8739E+06
1.015	3.5846E+01	1.0004E+02	6.5429E+02	6.8702E+03	9.6857E+04	1.7049E+06
1.020	2.7010E+01	7.3878E+01	4.7375E+02	4.8684E+03	6.7083E+04	1.1531E+06
1.025	2.1701E+01	5.8248E+01	3.6681E+02	3.6966E+03	4.9897E+04	8.3968E+05
1.030	1.8156E+01	4.7873E+01	2.9642E+02	2.9339E+03	3.8860E+04	6.4134E+05
1.035	1.5620E+01	4.0494E+01	2.4676E+02	2.4017E+03	3.1255E+04	5.0661E+05
1.040	1.3715E+01	3.4984E+01	2.0998E+02	2.0116E+03	2.5749E+04	4.1036E+05
1.044	1.2502E+01	3.1491E+01	1.8683E+02	1.7683E+03	2.2352E+04	3.5168E+05
1.077	7.2790E+00	1.6722E+01	9.1171E+01	7.9221E+02	9.1646E+03	1.3176E+05
2.000	5.0826E-01	7.0854E-01	1.0741E+00	3.5187E+00	1.6252E+01	9.2329E+01
2.250	3.2591E-01	5.6468E-01	6.9783E-01	1.8346E+00	7.3783E+00	3.6591E+01
2.500	1.8309E-01	4.6753E-01	5.1469E-01	1.0683E+00	3.7667E+00	1.6602E+01
2.750	6.6784E-02	3.8952E-01	4.1750E-01	6.8371E-01	2.0999E+00	8.3329E+00
3.000	-2.8402E-02	3.1915E-01	3.6093E-01	4.7813E-01	1.2567E+00	4.5253E+00
3.250	-1.0475E-01	2.5172E-01	3.2331E-01	3.6390E-01	8.0004E-01	2.6190E+00
3.500	-1.6331E-01	1.8580E-01	2.9346E-01	2.9895E-01	5.4015E-01	1.5983E+00
3.750	-2.0476E-01	1.2162E-01	2.6536E-01	2.6127E-01	3.8722E-01	1.0213E+00
4.000	-2.2983E-01	6.0270E-02	2.3588E-01	2.3845E-01	2.9555E-01	6.8079E-01
4.250	-2.3951E-01	3.2446E-03	2.0356E-01	2.2307E-01	2.4033E-01	4.7299E-01
4.500	-2.3517E-01	-4.7873E-02	1.6809E-01	2.1050E-01	2.0726E-01	3.4323E-01
4.750	-2.1852E-01	-9.1602E-02	1.2988E-01	1.9783E-01	1.8766E-01	2.6115E-01
5.000	-1.9162E-01	-1.2671E-01	8.9864E-02	1.8325E-01	1.7594E-01	2.0916E-01
6.000	-2.9775E-02	-1.6979E-01	-6.1244E-02	9.4146E-02	1.4811E-01	1.3792E-01
7.000	1.0606E-01	-8.0717E-02	-1.3591E-01	-2.2914E-02	9.0542E-02	1.2317E-01
8.000	1.2402E-01	4.6202E-02	-1.0045E-01	-1.0390E-01	-7.8107E-05	8.4635E-02
9.000	3.8831E-02	1.0923E-01	3.8637E-04	-1.0196E-01	-7.7581E-02	1.3901E-02
10.000	-6.0512E-02	7.3153E-02	8.1580E-02	-2.9194E-02	-9.5346E-02	-5.6743E-02

REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704-01-0188*

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From - To)*
September 2008	Final	

4. TITLE AND SUBTITLE

THE MODAL DECOMPOSITION OF THE QUALITY FACTOR OF AN ANTENNA IN PROLATE SPHEROIDAL COORDINATES

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHORS

R. C. Adams
P. M. Hansen

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

SSC San Diego
San Diego, CA 92152–5001

8. PERFORMING ORGANIZATION REPORT NUMBER

TR 1969

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Space and Naval Warfare Systems Command (PMW 770-2)
San Diego CA 92110-3127

10. SPONSOR/MONITOR'S ACRONYM(S)

SPAWAR

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

Many SSC San Diego public release documents are available in electronic format at
http://www.spawar.navy.mil/sti/publications/pubs/index.html

14. ABSTRACT

This report discusses extending the results for prolate spheroids by including higher order modes and finding the sum of modes giving the minimum radiation Q. There is no coupling between odd and even modes, but there is coupling between the lowest order even modes. The report also presents a simple result for the values of the degree of mixing of modes 2, 4, and 6 that minimizes Q_r.

15. SUBJECT TERMS

Mission Area: Command, Control, Communications

modal decomposition	dipole antennas	lower bound	electrically small antenna
prolate spheroidal coordinates	aspect ratio	radiation Q	cylinder

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			R. C. Adams
					19B. TELEPHONE NUMBER *(Include area code)*
U	U	U	UU	61	(619) 553–4313

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

INITIAL DISTRIBUTION

84300	Library	(2)
85300	Archive/Stock	(1)
85300	S. Baxley	(1)
55405	P. M. Hansen	(5)
5542	R. C. Adams	(8)

Defense Technical Information Center
Fort Belvoir, VA 22060–6218 (1)

SSC San Diego Liaison Office
C/O PEO-SCS
Arlington, VA 22202–4804 (1)

Center for Naval Analyses
Alexandria, VA 22311–1850 (1)

Government-Industry Data Exchange
Program Operations Center
Corona, CA 91718–8000 (1)